

# ASYMPTOTIC BEHAVIORS OF FUNDAMENTAL SOLUTION AND ITS DERIVATIVES RELATED TO SPACE-TIME FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Let  $p(t, x)$  be the fundamental solution to the problem

$$\partial_t^\alpha u = -(-\Delta)^\beta u, \quad \alpha \in (0, 2), \beta \in (0, \infty).$$

In this paper we provide the asymptotic behaviors and sharp upper bounds of  $p(t, x)$  and its space and time fractional derivatives

$$D_x^n (-\Delta_x)^\gamma D_t^\sigma I_t^\delta p(t, x), \quad \forall n \in \mathbb{Z}_+, \gamma \in [0, \beta], \sigma, \delta \in [0, \infty),$$

where  $D_x^n$  is a partial derivative of order  $n$  with respect to  $x$ ,  $(-\Delta_x)^\gamma$  is a fractional Laplace operator and  $D_t^\sigma$  and  $I_t^\delta$  are Riemann-Liouville fractional derivative and integral respectively.

## 1. INTRODUCTION

Let  $\alpha \in (0, 2)$ ,  $\beta \in (0, \infty)$  and  $p(t, x)$  be the fundamental solution to the space-time fractional equation

$$\partial_t^\alpha u = \Delta^\beta u, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d \quad (1.1)$$

with  $u(0, x) = u_0(x)$  (and  $\partial_t u(0, x) = 0$  if  $\alpha > 1$ ). Here  $\partial_t^\alpha$  denotes Caputo fractional derivative and  $\Delta^\beta := -(-\Delta)^\beta$  is the fractional Laplacian. The fractional time derivative of order  $\alpha \in (0, 1)$  can be used to model the anomalous diffusion exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena (see [25, 28]), and the fractional spatial derivative describes long range jumps of particles. The fractional wave equation  $\partial_t^\alpha u = \Delta u$  with  $\alpha \in (1, 2)$  governs the propagation of mechanical diffusive waves in viscoelastic media (see [23, 33]). Equation (1.1) has been an important topic in the mathematical physics related to non-Markovian diffusion processes with a memory [26, 27], in the probability theory related to jump processes [5, 6, 24] and in the theory of differential equations [7, 8, 17, 31, 36].

The aim of this paper is to present rigorous and self-contained exposition of fundamental solution  $p(t, x)$ . More precisely, we provide asymptotic behaviors and upper bound of

$$D_x^n (-\Delta_x)^\gamma D_t^\sigma I_t^\delta p(t, x), \quad \forall n \in \mathbb{Z}_+, \gamma \in [0, \beta], \sigma, \delta \in [0, \infty), \quad (1.2)$$

where  $I_t^\delta$  and  $D_t^\sigma$  denotes the Riemann-Liouville fractional integral and derivative respectively.

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There are certainly considerable works dealing with explicit formula for the fundamental solutions and their asymptotic behaviors (see e.g. [3, 9, 10, 11, 12, 13, 20, 22]). However, only few of them cover the derivative estimates of the fundamental solutions. In [9, 10, 17, 20], upper bounds of (1.2) were obtained for  $\beta = 1$ ,  $\sigma = 1 - \alpha$ , and  $\gamma = 0$ . Also, in [13] asymptotic behavior for the case  $\alpha = 1, \beta \in (0, 1)$  and  $\sigma = 0$  was obtained. Note that it is assumed that either  $\alpha = 1$  or  $\beta = 1$  in [9, 10, 13, 17, 20], and moreover spatial fractional derivative  $\Delta^\gamma p$  and time fractional derivative  $D_t^\sigma p$  are not obtained in [9, 10, 17, 20] and [13] respectively. Our result substantially improves these results because we only assume  $\alpha \in (0, 2)$  and  $\beta \in (0, \infty)$  and we provide two sides estimates of both space and time fractional derivatives of arbitrary order.

Our approach relies also on the properties of some special functions including the Fox H functions. However, unlike most of previous works, we do not use the method of Mellin, Laplace and inverse Fourier transforms which, due to the non-exponential decay at infinity of the Fox H functions, require restrictions on the space dimension  $d$  and other parameters  $\beta, \delta, \gamma$ , and  $\sigma$  (see Remark 5.2).

Below we give two main applications of our results. First, consider the non-homogeneous fractional evolution equation

$$\partial_t^\alpha u = \Delta^\beta u + f, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d \quad (1.3)$$

with  $u(0, x) = 0$  and additionally  $\partial_t u(0, x) = 0$  if  $\alpha > 1$ . One can show (see e.g. [9, 17, 20]) that the solution to the problem is given by

$$\mathcal{G}f(t, x) = \int_0^t \int_{\mathbb{R}^d} Q_\alpha(t-s, x-y) f(s, y) dy ds$$

where

$$Q_\alpha(t, x) := \begin{cases} D_t^{1-\alpha} p(t, x) & : \alpha \in (0, 1) \\ I_t^{\alpha-1} p(t, x) & : \alpha \in (1, 2). \end{cases}$$

It turns out (see [18, 19]) that to obtain the  $L_q(L_p)$ -estimate

$$\|\Delta^\beta \mathcal{G}f\|_{L_q((0, T), L_p(\mathbb{R}^d))} \leq N \|f\|_{L_q((0, T), L_p(\mathbb{R}^d))}, \quad p, q > 1 \quad (1.4)$$

it is sufficient to show that for  $a, b > 0$

$$\begin{aligned} \int_0^a \int_{|x| \geq b} |\Delta^\beta Q_\alpha(t, x)| dx dt &\leq N \frac{a^\alpha}{b^{2\beta}}, \\ \int_a^\infty \int_{\mathbb{R}^d} |\partial_t \Delta^\beta Q_\alpha(t, x)| dx dy &\leq \frac{N}{a}, \quad \int_a^\infty \int_{\mathbb{R}^d} |\nabla_x \Delta^\beta Q_\alpha(t, x)| dx dt \leq N a^{-\frac{\alpha}{2\beta}}. \end{aligned}$$

One can use our estimates to prove the above three inequalities, and therefore (1.4) can be obtained as a corollary. Our second application is the  $L_p$ -theory of the stochastic partial differential equations of the type

$$\partial_t^\alpha u = \Delta u + \partial_t^{\alpha+\sigma} \int_0^t g(s, x) dW_s,$$

where  $\sigma < \frac{1}{2}$  and  $W_t$  is a Wiener process defined on a probability space  $(\Omega, dP)$ . One can show (see [5]) that the solution to this problem is given by the formula

$$u(t, x) = \int_0^t \left( \int_{\mathbb{R}^d} P_\sigma(t-s, x-y) g(s, y) dy \right) dW_s.$$

Here  $P_\sigma(t, x)$  is defined as

$$P_\sigma(t, x) := \begin{cases} I_t^{|\sigma|} p(t, x) & : \sigma \leq 0 \\ D_t^{|\sigma|} p(t, x) & : \sigma > 0. \end{cases}$$

As has been shown for the case  $\alpha = 1$  (see [15, 16, 21, 35]), sharp estimates of  $D_x^n \Delta^\gamma P_\sigma$  can be used to obtain  $L_p$ -estimate

$$\|\Delta^\gamma u\|_{L_p(\Omega \times (0, T) \times \mathbb{R}^d)} \leq N \|g\|_{L_p(\Omega \times (0, T) \times \mathbb{R}^d)}. \quad (1.5)$$

The detail of (1.5) will be given for  $\gamma \leq (2 \wedge \frac{1-2\sigma}{\alpha})$  in a subsequent paper.

The rest of the article is organized as follows. In Section 2 we state our main results, Theorems 2.1, 2.3, and 2.4. In Section 3 we present the definition of the Fox H functions and their several properties. For the convenience of the reader, we repeat the relevant material and demonstration in [14] and [17], thus making our exposition self-contained. Section 4 contains asymptotic behaviors at zero and infinity of the Fox H function. In Section 5 we present explicit representation of fundamental solutions and their fractional and classical derivatives. Finally, in Section 6 we prove our main results.

We finish the introduction with some notion used in this article. We write  $f \lesssim g$  for  $|x| \leq \delta$  (resp.  $|x| \geq \delta$ ) if there exists a positive constant  $C$  independent of  $x$  such that  $f(x) \leq Cg(x)$  for  $|x| \leq \delta$  (resp.  $|x| \geq \delta$ ), and  $f \sim g$  for  $|x| \leq \delta$  (resp.  $|x| \geq \delta$ ) if  $f \lesssim g \lesssim f$  for  $|x| \leq \delta$  (resp.  $|x| \geq \delta$ ). We say  $f \sim g$  as  $|x| \rightarrow 0$  (resp.  $|x| \rightarrow \infty$ ) if there exists  $\varepsilon \in (0, 1)$  such that  $f \sim g$  for  $|x| \leq \varepsilon$  (resp.  $|x| \geq \varepsilon^{-1}$ ). We write  $f(x) = O(g(|x|))$  as  $|x| \rightarrow 0$  (resp.  $|x| \rightarrow \infty$ ) if there exists  $\delta > 0$  such that  $|f(x)| \lesssim |g(|x|)|$  for  $|x| < \delta$  (resp.  $|x| \geq \delta$ ). We use “:=” to denote a definition. As usual  $\mathbb{R}^d$  stands for the Euclidean space of points  $x = (x^1, \dots, x^d)$ ,  $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$ , and  $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ . For multi-indices  $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}_+^d$ ,  $n \in \mathbb{N}$  and functions  $u(x)$  we set

$$D_i u = \frac{\partial u}{\partial x^i}, \quad D_x^{\mathbf{a}} u = D_1^{a_1} \cdots D_d^{a_d} u, \quad D_x^n := \{D_x^{\mathbf{a}} : |\mathbf{a}| = n\}.$$

$[a]$  is the biggest integer which is less than or equal to  $a$ . By  $\mathcal{F}$  we denote the  $d$ -dimensional Fourier transform, that is,

$$\mathcal{F}\{f\}(\xi) := \int_{\mathbb{R}^d} e^{-i(x, \xi)} f(x) dx.$$

For a complex number  $z$ ,  $\Re[z]$  and  $\Im[z]$  are the real part and imaginary part of  $z$  respectively.

## 2. MAIN RESULTS

We first introduce some definitions related to the fractional calculus. Let  $\beta \geq 0$ . For a function  $u \in L_1(\mathbb{R}^d)$ , we write  $\Delta^\beta u = f$  if there exists a function  $f \in L_1(\mathbb{R}^d)$  such that

$$\mathcal{F}\{f(\cdot)\}(\xi) = |\xi|^{2\beta} \mathcal{F}\{u\}(\xi).$$

For  $u \in L_1((0, T))$ , the Riemann-Liouville fractional integral of the order  $\alpha \in (0, \infty)$  is defined as

$$I_t^\alpha u := \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t \leq T.$$

One can easily check

$$I_t^\alpha I_t^\beta = I_t^{\alpha+\beta}, \quad \forall \alpha, \beta \geq 0. \quad (2.1)$$

Let  $n \in \mathbb{N}$  and  $n - 1 \leq \alpha < n$ . The Riemann-Liouville fractional derivative  $D_t^\alpha$  and the Caputo fractional derivative  $\partial_t^\alpha$  are defined as

$$D_t^\alpha u := \left( \frac{d}{dt} \right)^n (I_t^{n-\alpha} u) \quad (2.2)$$

$$\partial_t^\alpha u := D_t^{\alpha-(n-1)} \left( u^{(n-1)}(t) - u^{(n-1)}(0) \right).$$

By definition (2.2) for any  $\alpha \geq 0$  and  $u \in L_1((0, T))$ ,

$$D_t^\alpha I_t^\alpha u = u. \quad (2.3)$$

Using (2.1)-(2.3), one can check

$$\partial_t^\alpha u = D_t^\alpha \left( u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) \right), \quad n-1 \leq \alpha < n.$$

Thus  $D_t^\alpha u = \partial_t^\alpha u$  if  $u(0) = u^{(1)}(0) = \dots = u^{(n-1)}(0) = 0$ . For more information on the fractional derivatives, we refer the reader to [29, 32].

For  $\sigma \in \mathbb{R}$  we define Riemann-Liouville fractional operator  $\mathbb{D}_t^\sigma$  as

$$\mathbb{D}_t^\sigma := \begin{cases} D_t^{|\sigma|} & : \sigma > 0 \\ I_t^{|\sigma|} & : \sigma < 0. \end{cases}$$

Then by (2.1) and (2.2), for any  $\alpha, \beta \geq 0$

$$D_t^\alpha I_t^\beta = \mathbb{D}_t^{\alpha-\beta}.$$

The Mittag-Leffler function  $E_{\alpha,\beta}(z)$  is defined as

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \Re[\alpha] > 0 \quad (2.4)$$

for  $z \in \mathbb{C}$  and we write  $E_\alpha(z) = E_{\alpha,1}(z)$  for short. Using the equality

$$\mathbb{D}_t^\sigma t^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\sigma)} t^{\alpha-\sigma}, \quad \forall \sigma \in \mathbb{R}, \alpha \geq 0$$

one can check that for  $t > 0$

$$\mathbb{D}_t^\sigma E_\alpha(-\lambda t^\alpha) = t^{-\sigma} E_{\alpha,1-\sigma}(-\lambda t^\alpha), \quad \sigma \in \mathbb{R}, \lambda \geq 0,$$

and that for any constant  $\lambda$ ,

$$\varphi(t) := E_\alpha(-\lambda t^\alpha)$$

satisfies  $\varphi(0) = 1$  (also  $\varphi'(0) = 0$  if  $\alpha > 1$ ) and

$$\partial_t^\alpha \varphi = -\lambda \varphi, \quad t > 0.$$

Let  $\alpha \in (0, 2)$  and  $\beta \in (0, \infty)$ . By taking the Fourier transform to the equation

$$\partial_t^\alpha u = \Delta^\beta u, \quad t > 0, \quad u(0, x) = u_0(x), \quad (\text{and } u'(0, x) = 0 \text{ if } \alpha > 1)$$

one can formally get

$$\mathcal{F}\{u(t, \cdot)\} = E_\alpha(-|\xi|^{2\beta} t^\alpha) \mathcal{F}\{u_0\}.$$

Therefore, to obtain the fundamental solution, it is needed to find an integrable function  $p(t, x) \in L_1(\mathbb{R}^d)$  satisfying

$$\mathcal{F}\{p(t, \cdot)\} = E_\alpha(-|\xi|^{2\beta} t^\alpha). \quad (2.5)$$

Denote

$$\mathbf{M}(t, x) := |x|^{2\beta} t^{-\alpha}.$$

In the following theorems we give the asymptotic behaviors of  $D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t, x)$  as  $\mathbf{M} \rightarrow 0$  and  $\mathbf{M} \rightarrow \infty$ . We also provide upper bounds when  $\mathbf{M} \leq 1$  and  $\mathbf{M} \geq 1$ .

Firstly, we consider the case  $\mathbf{M} \rightarrow \infty$ .

**Theorem 2.1.** *Let  $\alpha \in (0, 2)$ ,  $\beta \in (0, \infty)$ ,  $\gamma \in [0, \infty)$ ,  $\sigma \in \mathbb{R}$  and  $n \in \mathbb{N}$ . There exists a function  $p(t, \cdot) \in L_1(\mathbb{R}^d)$  satisfying (2.5). Furthermore, the following asymptotic behaviors hold as  $\mathbf{M} \rightarrow \infty$ :*

(i) *If  $\beta \in \mathbb{N}$ , then for some constant  $c > 0$  depending only on  $d, n, \alpha, \beta, \sigma$*

$$|\mathbb{D}_t^\sigma p(t, x)| \lesssim |x|^{-d} t^{-\sigma} \exp \left\{ -c |x|^{\frac{2\beta}{2\beta-\alpha}} t^{-\frac{\alpha}{2\beta-\alpha}} \right\}$$

and

$$|D_x^n \mathbb{D}_t^\sigma p(t, x)| \lesssim |x|^{-d-n} t^{-\sigma} \exp \left\{ -c |x|^{\frac{2\beta}{2\beta-\alpha}} t^{-\frac{\alpha}{2\beta-\alpha}} \right\}. \quad (2.6)$$

(ii) *If  $\alpha = 1$ ,  $\beta \notin \mathbb{N}$ , and  $\sigma = 0$ ,*

$$|\Delta^\gamma p(t, x)| \sim \begin{cases} t |x|^{-d-2\gamma-2\beta} & : \gamma \in \mathbb{Z}_+ \\ |x|^{-d-2\gamma} & : \gamma \in [0, \infty) \setminus \mathbb{Z}_+ \end{cases} \quad (2.7)$$

and

$$|D_x^n \Delta^\gamma p(t, x)| \lesssim \begin{cases} t |x|^{-d-2\gamma-2\beta-n} & : \gamma \in \mathbb{Z}_+ \\ |x|^{-d-2\gamma-n} & : \gamma \in [0, \infty) \setminus \mathbb{Z}_+. \end{cases}$$

(iii) *If  $\gamma \in (0, \beta) \setminus \mathbb{N}$ ,*

$$|\Delta^\gamma \mathbb{D}_t^\sigma p(t, x)| \sim |x|^{-d-2\gamma} t^{-\sigma}$$

and

$$|D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t, x)| \lesssim |x|^{-d-2\gamma-n} t^{-\sigma}.$$

(iv) *If  $\beta \notin \mathbb{N}$  and  $\gamma \in [0, \beta) \cap \mathbb{Z}_+$ ,*

$$|\Delta^\gamma \mathbb{D}_t^\sigma p(t, x)| \sim |x|^{-d-2\gamma-2\beta} t^{-\sigma+\alpha} \quad (2.8)$$

and

$$|D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t, x)| \lesssim |x|^{-d-2\gamma-2\beta-n} t^{-\sigma+\alpha}.$$

(v) *If  $\gamma = \beta$  and  $d \geq 2$ ,*

$$|\Delta^\beta \mathbb{D}_t^\sigma p(t, x)| \sim \begin{cases} |x|^{-d-4\beta} t^{-\sigma+\alpha} & : \beta \in \mathbb{N} \text{ or } \sigma \in \mathbb{N} \\ |x|^{-d-2\beta} t^{-\sigma} & : \text{otherwise} \end{cases} \quad (2.9)$$

and

$$|D_x^n \Delta^\beta \mathbb{D}_t^\sigma p(t, x)| \lesssim \begin{cases} |x|^{-d-4\beta-n} t^{-\sigma+\alpha} & : \beta \in \mathbb{N} \text{ or } \sigma \in \mathbb{N} \\ |x|^{-d-2\beta-n} t^{-\sigma} & : \text{otherwise.} \end{cases} \quad (2.10)$$

(vi) *If  $\gamma = \beta$ ,  $\sigma + \alpha \in \mathbb{N}$ , and  $d = 1$ , then (2.9) and (2.10) hold.*

*Remark 2.2.* (i) Note that  $D_x^n \mathbb{D}_t^\sigma p(t, x)$  has exponential decay as  $\mathbf{M} \rightarrow \infty$  only when  $\beta$  is a positive integer.

(ii) Let  $x \neq 0$ . Then by (2.6) and (2.8) with  $\gamma = 0$ ,  $D_x^n \mathbb{D}_t^\sigma p(t, x) \rightarrow 0$  as  $t \rightarrow 0$  if either  $\beta$  is a positive integer or  $\sigma < \alpha$ .

(iii) If  $\gamma = \beta$  and  $d = 1$ , then we additionally assumed  $\sigma + \alpha \in \mathbb{N}$ . Without this extra condition we had a trouble in using Fubini's theorem in our proof.

(iv) Note that we have only upper bounds of  $D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t, x)$  unless  $n = 0$ . This is because, for instance,  $D_i p(t, x)$  is of type  $x^i g(t, x)$  (see (5.1)) and becomes zero if  $x^i = 0$ . Hence we can not have positive lower bound of  $D_i p(t, x)$  for such  $x$ .

Secondly, we consider the case  $\mathbf{M} \rightarrow 0$ .

**Theorem 2.3.** *Let  $\alpha, \beta, \gamma, \sigma$  be given as in Theorem 2.1 and  $n \in \mathbb{N}$ . Then the following asymptotic behaviors hold as  $\mathbf{M} \rightarrow 0$ :*

(i) *If  $\gamma \in [0, \beta)$  and  $\sigma + \alpha \notin \mathbb{N}$ ,*

$$|\Delta^\gamma \mathbb{D}_t^\sigma p(t, x)| \sim \begin{cases} t^{-\sigma - \frac{\alpha(d+2\gamma)}{2\beta}} & : \gamma < \beta - \frac{d}{2} \\ |x|^{-d-2\gamma+2\beta} t^{-\sigma-\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \gamma = \beta - \frac{d}{2} \\ |x|^{-d-2\gamma+2\beta} t^{-\sigma-\alpha} & : \gamma > \beta - \frac{d}{2} \end{cases} \quad (2.11)$$

and

$$|D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t, x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma - \frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma < \beta - \frac{d}{2} - 1 \\ |x|^{2-n} t^{-\sigma-\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \gamma = \beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma+2\beta-n} t^{-\sigma-\alpha} & : \gamma > \beta - \frac{d}{2} - 1. \end{cases} \quad (2.12)$$

(ii) *If  $\gamma \in [0, \beta)$  and  $\sigma + \alpha \in \mathbb{N}$ ,*

$$|\Delta^\gamma \mathbb{D}_t^\sigma p(t, x)| \sim \begin{cases} t^{-\sigma - \frac{\alpha(d+2\gamma)}{2\beta}} & : \gamma < 2\beta - \frac{d}{2} \\ |x|^{-d-2\gamma+4\beta} t^{-\sigma-2\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \gamma = 2\beta - \frac{d}{2} \\ |x|^{-d-2\gamma+4\beta} t^{-\sigma-2\alpha} & : \gamma > 2\beta - \frac{d}{2} \end{cases} \quad (2.13)$$

and

$$|D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t, x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma - \frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma < 2\beta - \frac{d}{2} - 1 \\ |x|^{2-n} t^{-\sigma-2\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \gamma = 2\beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma+4\beta-n} t^{-\sigma-2\alpha} & : \gamma > 2\beta - \frac{d}{2} - 1. \end{cases} \quad (2.14)$$

(iii) *If  $\alpha = 1$  and  $\sigma = 0$ ,*

$$|\Delta^\gamma p(t, x)| \sim t^{-\frac{d+2\gamma}{2\beta}}, \quad |D_x^n \Delta^\gamma p(t, x)| \lesssim |x|^{2-n} t^{-\frac{d+2\gamma+2}{2\beta}}.$$

(iv) *If  $\gamma = \beta$  and  $d \geq 2$ ,*

$$|\Delta^\beta \mathbb{D}_t^\sigma p(t, x)| \sim \begin{cases} t^{-\sigma-\alpha - \frac{\alpha d}{2\beta}} & : \frac{d}{2} < \beta \\ |x|^{-d+2\beta} t^{-\sigma-2\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \frac{d}{2} = \beta \\ |x|^{-d+2\beta} t^{-\sigma-2\alpha} & : \frac{d}{2} > \beta \end{cases} \quad (2.15)$$

and

$$|D_x^n \Delta^\beta \mathbb{D}_t^\sigma p(t, x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma-\alpha - \frac{\alpha(d+2)}{2\beta}} & : \frac{d}{2} + 1 < \beta \\ |x|^{2-n} t^{-\sigma-2\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \frac{d}{2} + 1 = \beta \\ |x|^{-d+2\beta-n} t^{-\sigma-2\alpha} & : \frac{d}{2} + 1 > \beta. \end{cases} \quad (2.16)$$

(v) If  $\gamma = \beta$ ,  $\sigma + \alpha \in \mathbb{N}$ , and  $d = 1$ , then (2.15) and (2.16) hold.

Next we give the upper estimates when  $\mathbf{M} \geq 1$  and  $\mathbf{M} \leq 1$ .

**Theorem 2.4.** (i) Assertions (i)-(iv) of Theorem 2.1 also hold for  $\mathbf{M} \geq 1$  if “ $\sim$ ” is replaced by “ $\lesssim$ ”.

(ii) Assertions (i)-(v) of Theorem 2.3 also hold for  $\mathbf{M} \leq 1$  if “ $\sim$ ” is replaced by “ $\lesssim$ ”.

The proofs of Theorems 2.1, 2.3 and 2.4 are given in Section 6.

*Remark 2.5.* Let  $\alpha = 1$  and  $\beta \in (0, \infty)$ , and  $\sigma = 0$ . Then by Theorem 2.4,

$$|D_x^n \Delta^\gamma p(t, x)| \lesssim \begin{cases} t|x|^{-d-2\beta-2\gamma-n} & : \gamma \in \mathbb{Z}_+ \\ |x|^{-d-2\gamma-n} & : \gamma \notin \mathbb{Z}_+ \end{cases}$$

holds for  $|x|^2 \geq t$ . Also, by Theorem 2.4, for  $|x|^2 \leq t$ ,

$$|\Delta^\gamma p(t, x)| \lesssim t^{-\frac{d+2\gamma}{2\beta}}, \quad |D_x^n \Delta^\gamma p(t, x)| \lesssim |x|^{2-n} t^{-\frac{d+2\gamma+2}{2\beta}}.$$

These estimates cover the results of [15, Lemma 3.1, 3.3] and [13, Corollary 1].

*Remark 2.6.* Theorem 2.4 also implies the results of [9, Proposition 5.1, 5.2]. Let  $\beta = 1$  and take a  $|n| \geq 1$ . For  $|x|^2 \geq t^\alpha$ .

$$\begin{aligned} |D_x^n \mathbb{D}_t^\sigma p(t, x)| &\lesssim |x|^{-d-n} t^{-\sigma} \exp \left\{ -ct^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{2}{2-\alpha}} \right\}. \\ &\lesssim t^{-\frac{\alpha(d+n)}{2}-\sigma} \exp \left\{ -ct^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{2}{2-\alpha}} \right\}. \end{aligned}$$

Also, (cf. (2.11), (2.12), and (2.14)),

$$|p(t, x)| \lesssim \begin{cases} t^{-\frac{\alpha d}{2}} & : d \geq 3 \\ t^{-\alpha} (1 + |\ln |x|^2 t^{-\alpha}|) & : d = 2 \\ |x|^{-d+2} t^{-\alpha} & : d = 1 \end{cases}$$

and

$$|D_x^n p(t, x)| \lesssim |x|^{-d+2-n} t^{-\alpha}, \quad |D_x^n \Delta p(t, x)| \lesssim |x|^{-d+2-n} t^{-2\alpha},$$

$$|D_x^n \mathbb{D}_t^{1-\alpha} p(t, x)| \lesssim \begin{cases} |x|^{-d+4-n} t^{-\alpha-1} & : d \geq 3 \\ |x|^{2-n} t^{-1} (1 + |\ln |x|^2 t^{-\alpha}|) & : d = 2 \\ |x|^{1-n} t^{-1} & : d = 1 \end{cases}$$

hold for  $|x|^2 \leq t^\alpha$ .

### 3. THE FOX H FUNCTION

In this section, we introduce the definition and some properties of the Fox H function. We refer to [14] for further information.

**3.1. Definition.** Let  $\Gamma(z)$  denote the gamma function which can be defined (see [2, Section 1.1]) for  $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$  as

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!n^z}{z(z+1) \cdots (z+n)}.$$

Note that  $\Gamma(z)$  is a meromorphic function with simple poles at the nonpositive integers. From the definition, for  $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ ,

$$z\Gamma(z) = \Gamma(z+1), \quad (3.1)$$

and it holds that

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}, \quad \prod_{k=0}^{m-1} \Gamma(z + \frac{k}{m}) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mz} \Gamma(mz). \quad (3.2)$$

One can easily check that for  $k \in \mathbb{Z}_+$ ,

$$\text{Res}_{z=-k}[\Gamma(z)] = \lim_{z \rightarrow -k} (z+k)\Gamma(z) = \frac{(-1)^k}{k!},$$

where  $\text{Res}_{z=z_0}[f(z)]$  denotes the residue of  $f(z)$  at  $z = z_0$ . From the Stirling's approximation

$$\Gamma(z) \sim \sqrt{2\pi} e^{(z-\frac{1}{2}) \log z} e^{-z}, \quad |z| \rightarrow \infty,$$

it follows that

$$|\Gamma(a+ib)| \sim \sqrt{2\pi} |a|^{a-\frac{1}{2}} e^{-a-\pi[1-\text{sign}(a)]b/2}, \quad |a| \rightarrow \infty \quad (3.3)$$

and

$$|\Gamma(a+ib)| \sim \sqrt{2\pi} |b|^{a-\frac{1}{2}} e^{-a-\frac{\pi|b|}{2}}, \quad |b| \rightarrow \infty. \quad (3.4)$$

Let  $m, n, \nu, \mu$  be fixed integers satisfying  $0 \leq m \leq \mu$ ,  $0 \leq n \leq \nu$ . Assume that real parameters  $\mathbf{c}_1, \dots, \mathbf{c}_\nu$ ,  $\mathbf{d}_1, \dots, \mathbf{d}_\mu$  and positive real parameters  $\gamma_1, \dots, \gamma_\nu$ ,  $\delta_1, \dots, \delta_\mu$  are given such that

$$\max_{1 \leq j \leq m} \left( -\frac{\mathbf{d}_j}{\delta_j} \right) < \min_{1 \leq j \leq n} \left( \frac{1 - \mathbf{c}_j}{\gamma_j} \right). \quad (3.5)$$

For each  $k \in \mathbb{Z}_+$ , we set

$$\mathbf{c}_{j,k} = \frac{1 - \mathbf{c}_j + k}{\gamma_j}, \quad \mathbf{d}_{j,k} = -\frac{\mathbf{d}_j + k}{\delta_j},$$

which constitute

$$P_1 = \{\mathbf{d}_{j,k} \in \mathbb{R} : j \in \{1, \dots, m\}, k \in \mathbb{Z}_+\},$$

$$P_2 = \{\mathbf{c}_{j,k} \in \mathbb{R} : j \in \{1, \dots, n\}, k \in \mathbb{Z}_+\}.$$

Note that  $P_1 \cap P_2 = \emptyset$  by (3.5). We arrange the elements of  $P_1$  and  $P_2$  as follows:

$$P_1 = \{\hat{\mathbf{d}}_0 > \hat{\mathbf{d}}_1 > \hat{\mathbf{d}}_2 > \dots\}, \quad P_2 = \{\hat{\mathbf{c}}_0 < \hat{\mathbf{c}}_1 < \hat{\mathbf{c}}_2 < \dots\}. \quad (3.6)$$

For the above parameters, define

$$\mathcal{H}(z) := \frac{\prod_{j=1}^m \Gamma(\mathbf{d}_j + \delta_j z) \prod_{j=1}^n \Gamma(1 - \mathbf{c}_j - \gamma_j z)}{\prod_{j=n+1}^\nu \Gamma(\mathbf{c}_j + \gamma_j z) \prod_{j=m+1}^\mu \Gamma(1 - \mathbf{d}_j - \delta_j z)}.$$



Note that  $P_1$  and  $P_2$  are sets of poles of  $\mathcal{H}(z)$ . To describe the behavior of  $\mathcal{H}(z)$  as  $|z| \rightarrow \infty$ , we set

$$\alpha^* := \sum_{i=1}^n \gamma_i - \sum_{i=n+1}^{\nu} \gamma_i + \sum_{j=1}^m \delta_j - \sum_{j=m+1}^{\mu} \delta_j,$$

and

$$\Lambda := \sum_{j=1}^{\mu} \mathfrak{d}_j - \sum_{j=1}^{\nu} \mathfrak{c}_j + \frac{\nu - \mu}{2}, \quad \omega := \sum_{j=1}^{\mu} \delta_j - \sum_{j=1}^{\nu} \gamma_i, \quad \eta := \prod_{j=1}^{\nu} \gamma_j^{-\gamma_j} \prod_{j=1}^{\mu} \delta_j^{\delta_j}.$$

Due to (3.3),

$$|\mathcal{H}(a + ib)r^{-a-ib}| \sim \left(\frac{e}{a}\right)^{-\omega a} \left(\frac{\eta}{r}\right)^a a^{\Lambda}, \quad r \in (0, \infty) \quad (3.7)$$

as  $a \rightarrow \infty$  and

$$|\mathcal{H}(a + ib)r^{-a-ib}| \sim \left(\frac{e}{|a|}\right)^{\omega|a|} \left(\frac{r}{\eta}\right)^{-|a|} |a|^{\Lambda}, \quad r \in (0, \infty) \quad (3.8)$$

as  $a \rightarrow -\infty$ . By (3.4), it follows that

$$|\mathcal{H}(a + ib)r^{-a-ib}| \sim \left(\frac{e}{|b|}\right)^{-\omega a} \left(\frac{\eta}{r}\right)^a |b|^{\Lambda} e^{-\alpha^* |b| \pi/2}, \quad r \in (0, \infty) \quad (3.9)$$

as  $|b| \rightarrow \infty$ .

The Fox H function  $H_{\nu\mu}^{mn}(r)$  ( $r > 0$ ) is defined via Mellin-Barnes type integral in the form

$$\begin{aligned} H_{\nu\mu}^{mn}(r) &:= H_{\nu\mu}^{mn} \left[ r \left| \begin{matrix} [\mathfrak{c}, \gamma] \\ [\mathfrak{d}, \delta] \end{matrix} \right. \right] \\ &:= H_{\nu\mu}^{mn} \left[ r \left| \begin{matrix} (\mathfrak{c}_1, \gamma_1) & \cdots & (\mathfrak{c}_{\nu}, \gamma_{\nu}) \\ (\mathfrak{d}_1, \delta_1) & \cdots & (\mathfrak{d}_{\mu}, \delta_{\mu}) \end{matrix} \right. \right] := \frac{1}{2\pi i} \int_L \mathcal{H}(z) r^{-z} dz. \end{aligned} \quad (3.10)$$

In (3.10),  $L$  is the infinite contour which separates all the poles in  $P_1$  to the left and all the poles in  $P_2$  to the right of  $L$ . Precisely, we choose  $L$  as follows:

- (i) if  $\omega > 0$ , then  $L = L_{Ha}^-$ , which is a left loop situated in a horizontal strip (or left Hankel contour), runs from  $-\infty + ih_1$  to  $\ell + ih_1$ , and then to  $\ell + ih_2$  and finally terminates at the point  $-\infty + ih_2$  with  $-\infty < h_1 < 0 < h_2 < \infty$  and

$$\hat{\mathfrak{d}}_0 < \ell < \hat{\mathfrak{c}}_0, \quad (3.11)$$

- (ii) if  $\omega < 0$ , then  $L = L_{Ha}^+$ , which is a right loop situated in a horizontal strip (or right Hankel contour), runs from  $+\infty + ih_1$  to  $\ell + ih_1$ , and then to  $\ell + ih_2$  and finally terminates at the point  $+\infty + ih_2$  with  $-\infty < h_1 < 0 < h_2 < \infty$  and (3.11),
- (iii) if  $\omega = 0$ , then  $L = L_{Ha}^-$  and  $L = L_{Ha}^+$  and for  $r \in (0, \eta)$  and  $r \in (\eta, \infty)$  respectively.

The following proposition shows that integral (3.10) is well defined and independent of the choice of  $h_1, h_2$ , and  $\ell \in (\hat{\mathfrak{d}}_0, \hat{\mathfrak{c}}_0)$ .

**Proposition 3.1** ([14, Theorem 1.2]). *Assume (3.5) and choose the contour  $L$  as above. Then Mellin-Barnes integral (3.10) makes sense and it is an analytic function of  $r \in (0, \infty)$  and of  $r \in (0, \eta) \cup (\eta, \infty)$  if  $\omega \neq 0$  and  $\omega = 0$  respectively. Furthermore,*

(i) if  $\omega > 0$ , then

$$H_{\nu\mu}^{mn}(r) = \sum_{k=0}^{\infty} \text{Res}_{z=\hat{\mathfrak{d}}_k} [\mathcal{H}(z)r^{-z}] \quad (3.12)$$

(ii) if  $\omega < 0$ , then

$$H_{\nu\mu}^{mn}(r) = - \sum_{k=0}^{\infty} \text{Res}_{z=\hat{\mathfrak{c}}_k} [\mathcal{H}(z)r^{-z}] \quad (3.13)$$

(iii) if  $\omega = 0$ , then (3.12) and (3.13) hold for  $r \in (0, \eta)$  and  $r \in (\eta, \infty)$  respectively.

*Proof.* (i) Let  $\omega > 0$  and  $r \in (0, \infty)$ . Choose  $h_1 < 0$ ,  $h_2 > 0$ , and  $\ell \in \mathbb{R}$  so that (3.11) holds. Take a sufficiently large  $p \in \mathbb{Z}_+$  so that  $\hat{\mathfrak{d}}_p < 0$ . Then there exists a real number  $M = M(p) > 0$  such that

$$-\hat{\mathfrak{d}}_p < M < -\hat{\mathfrak{d}}_{p+1}. \quad (3.14)$$

Define a closed rectangular contour  $C^M$  which can be decomposed into four lines

$$C_M = L_1 \cup L_2 \cup L_3 \cup L_4$$

where

$$\begin{aligned} L_1 &:= \{z \in \mathbb{C} : \Re[z] = \ell, h_1 \leq \Im[z] \leq h_2\}, \\ L_2 &:= \{z \in \mathbb{C} : \Re[z] = -M, h_1 \leq \Im[z] \leq h_2\}, \\ L_3 &:= \{z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = h_1\}, \\ L_4 &:= \{z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = h_2\}. \end{aligned}$$

Note that  $\mathcal{H}(z)r^{-z}$  is a meromorphic function on  $z \in \mathbb{C} \setminus P_1 \cup P_2$  and

$$\int_{C^M} |\mathcal{H}(z)r^{-z}| |dz| < \infty. \quad (3.15)$$

Due to (3.8) ( $i = 1, 2$ )

$$\begin{aligned} \left| \int_{-\infty}^{-M} \mathcal{H}(t + ih_i) r^{-t - ih_i} dt \right| &\leq \int_{-\infty}^{-M} |\mathcal{H}(t + ih_i) r^{-t - ih_i}| dt \\ &\lesssim \int_M^{\infty} \left(\frac{e}{t}\right)^{\omega t} \left(\frac{r}{\eta}\right)^{-t} t^{\Lambda} dt < \infty, \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} \left| \int_{L_2} \mathcal{H}(z)r^{-z} dz \right| &= \left| \int_{h_1}^{h_2} \mathcal{H}(-M + it) r^{M - it} dt \right| \\ &\leq \int_{h_1}^{h_2} |\mathcal{H}(-M + it) r^{M - it}| dt \\ &\lesssim \left(\frac{e}{M}\right)^{\omega M} \left(\frac{r}{\eta}\right)^M M^{\Lambda} (h_2 - h_1) < \infty. \end{aligned} \quad (3.17)$$

By (3.15)-(3.17), integral (3.10) absolutely converges and makes sense with  $L = L_{\overline{H}_a}$ .

We next show (3.12). Note that (3.16) and (3.17) converge to 0 as  $M \rightarrow \infty$  and by the theory of residues,

$$\frac{1}{2\pi i} \int_{C_M} \mathcal{H}(z) r^{-z} dz = \sum_{k=0}^p \text{Res}_{z=\hat{\mathfrak{d}}_k} [\mathcal{H}(z) r^{-z}].$$

Thus our claim follows immediately if we prove the convergence of the residue expansion. Let  $q \in \mathbb{N}$  ( $q > p$ ) be arbitrarily given. Then we can choose a real number  $N = N(q) > 0$  satisfying (3.14) where  $p$  and  $M$  are replaced by  $q$  and  $N$  respectively. Set

$$C'_M := L'_1 \cup L_2 \cup L'_3 \cup L'_4$$

where

$$\begin{aligned} L'_1 &:= \{z \in \mathbb{C} : \Re[z] = -N, h_1 \leq \Im[z] \leq h_2\}, \\ L'_3 &:= \{z \in \mathbb{C} : -N \leq \Re[z] \leq -M, \Im[z] = h_1\}, \\ L'_4 &:= \{z \in \mathbb{C} : -N \leq \Re[z] \leq -M, \Im[z] = h_2\}. \end{aligned}$$

Observe that

$$\frac{1}{2\pi i} \int_{C'_M} \mathcal{H}(z) r^{-z} dz = \sum_{k=p+1}^q \text{Res}_{z=\hat{\mathfrak{d}}_k} [\mathcal{H}(z) r^{-z}]. \quad (3.18)$$

By replacing  $M$  by  $N$  in (3.16) and (3.17),

$$\lim_{N \rightarrow \infty} \left| \int_{L'_1} \mathcal{H}(z) r^{-z} dz \right| = \lim_{N, M \rightarrow \infty} \left| \int_{L'_3 \cup L'_4} \mathcal{H}(z) r^{-z} dz \right| = 0,$$

which implies

$$\lim_{p, q \rightarrow \infty} \sum_{k=p+1}^q \text{Res}_{z=\hat{\mathfrak{d}}_k} [\mathcal{H}(z) r^{-z}] = \lim_{N, M \rightarrow \infty} \frac{1}{2\pi i} \int_{C'_M} \mathcal{H}(z) r^{-z} dz = 0.$$

Thus (3.12) is proved.

(ii) The case  $\omega < 0$  is an analogue of the case  $\omega > 0$ . By (3.7), for sufficiently large  $M > 0$  ( $i = 1, 2$ )

$$\begin{aligned} \left| \int_M^\infty \mathcal{H}(t + ih_i) r^{-t - ih_i} dt \right| &\leq \int_M^\infty |\mathcal{H}(t + ih_i) r^{-t - ih_i}| dt \\ &\lesssim \int_M^\infty \left(\frac{e}{t}\right)^{-\omega t} \left(\frac{\eta}{r}\right)^t t^\Lambda dt < \infty, \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} \left| \int_{L_2} \mathcal{H}(z) r^{-z} dz \right| &= \left| \int_{h_1}^{h_2} \mathcal{H}(M + it) r^{M - it} dt \right| \\ &\leq \int_{h_1}^{h_2} |\mathcal{H}(M + it) r^{M - it}| dt \\ &\lesssim \left(\frac{e}{M}\right)^{-\omega M} \left(\frac{\eta}{r}\right)^M M^\Lambda (h_2 - h_1) < \infty. \end{aligned} \quad (3.20)$$

Note that both (3.19) and (3.20) converge to 0 as  $M \rightarrow \infty$ . Then we obtain our desired result by replacing  $\hat{\mathfrak{d}}_k$  and  $M$  in the proof of the case  $\omega > 0$  by  $\hat{\mathfrak{c}}_k$  and  $-M$  respectively.

(iii) Finally we consider  $\omega = 0$ . Note that (3.16) and (3.17) hold if  $r < \eta$ , consequently (3.12) follows. If  $r > \eta$ , then we have (3.19) and (3.20) which give (3.13) immediately. The proposition is proved.  $\square$

In the remainder of this section, we assume

$$\alpha^* > 0. \quad (3.21)$$

Under (3.21), we can choose a contour  $L = L_{Br}$  which is a vertical contour (or Bromwich contour) starting at the point  $\ell - i\infty$  and terminating at the point  $\ell + i\infty$  where  $\ell$  satisfies (3.11). Actually,  $H_{\nu\mu}^{mn}(r)$  does not depend on the choice of  $L$  due to the following proposition and it is an analytic function of  $r \in (0, \infty)$  (it is a holomorphic function of  $r \in \mathbb{C}$  in the sector  $|\arg r| < \frac{\omega\pi}{2}$ . See [14, Theorem 1.2.(iii)]).

**Proposition 3.2.** *Under (3.21), Mellin-Barnes integral (3.10) makes sense with  $L = L_{Br}$ . Furthermore, for  $r \in (0, \infty)$*

$$\frac{1}{2\pi i} \int_{L_{Br}} \mathcal{H}(z) r^{-z} dz = \frac{1}{2\pi i} \int_{L_{Ha}^-} \mathcal{H}(z) r^{-z} dz \quad (3.22)$$

if  $\omega > 0$ ,

$$\frac{1}{2\pi i} \int_{L_{Br}} \mathcal{H}(z) r^{-z} dz = \frac{1}{2\pi i} \int_{L_{Ha}^+} \mathcal{H}(z) r^{-z} dz \quad (3.23)$$

if  $\omega < 0$ . If  $\omega = 0$  then (3.22) and (3.23) hold for  $r < \eta$  and  $r > \eta$  respectively.

*Proof.* We only prove the case  $\omega \geq 0$ . The proof of the other case is almost same. Fix  $r \in (0, \infty)$  and

$$L = L_{Br} = \{z \in \mathbb{C} : \Re[z] = \ell\},$$

where  $\ell$  satisfies (3.11). Note that the relation

$$|\mathcal{H}(\ell + it) r^{-\ell - it}| \sim e^{-\omega} \eta^\ell r^{-\ell} |t|^{\omega\ell + \Lambda} \exp\left\{-\frac{|t|\alpha^*}{2}\pi\right\}$$

holds as  $|t| \rightarrow \infty$  uniformly in  $\ell$  by (3.9). Hence the contour integral

$$\frac{1}{2\pi i} \int_{L_{Br}} \mathcal{H}(z) r^{-z} dz$$

absolutely converges and makes sense. Due to Proposition 3.1, we only need to show that

$$\frac{1}{2\pi i} \int_{L_{Br}} \mathcal{H}(z) r^{-z} dz = \sum_{k=0}^{\infty} \text{Res}_{z=\delta_k} [\mathcal{H}(z) r^{-z}]. \quad (3.24)$$

Given (sufficiently large)  $p \in \mathbb{Z}_+$ , we take  $M$  as in the proof of Proposition 3.1. Define a closed rectangular contour  $C^M$  which can be decomposed into four lines

$$C^M = L_v^M \cup L_v^{-M} \cup L_h^{-M} \cup L_h^M$$

where

$$\begin{aligned} L_v^M &:= \{z \in \mathbb{C} : \Re[z] = \ell, |\Im[z]| \leq M\}, \\ L_v^{-M} &:= \{z \in \mathbb{C} : \Re[z] = -M, |\Im[z]| \leq M\}, \\ L_h^M &:= \{z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = M\}, \\ L_h^{-M} &:= \{z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = -M\}. \end{aligned}$$

Then by the theorem of residues,

$$\frac{1}{2\pi i} \int_{C^M} \mathcal{H}(z) r^{-z} dz = \sum_{k=0}^p \text{Res}_{z=\delta_k} [\mathcal{H}(z) r^{-z}]. \quad (3.25)$$

By (3.8), for any  $h \in \mathbb{R}$ ,

$$|\mathcal{H}(t + ih) r^{-t-ih}| \sim \left( \frac{e}{|t|} \right)^{\omega|t|} \left( \frac{r}{\eta} \right)^{|t|} |t|^\Lambda$$

as  $t \rightarrow -\infty$ . Thus

$$\begin{aligned} \lim_{M \rightarrow \infty} \int_{L_v^{-M}} |\mathcal{H}(z) r^{-z}| |dz| &= \lim_{M \rightarrow \infty} \int_{-M}^M |\mathcal{H}(-M + ih) r^{M-ih}| dh \\ &\leq \lim_{M \rightarrow \infty} \int_{-M}^M \left( \frac{e}{M} \right)^{\omega M} \left( \frac{r}{\eta} \right)^M M^\Lambda dt \\ &\leq \lim_{M \rightarrow \infty} 2 \left( \frac{e}{M} \right)^{\omega M} \left( \frac{r}{\eta} \right)^M M^{\Lambda+1} = 0. \end{aligned}$$

On the other hand, by (3.9),

$$\begin{aligned} \lim_{M \rightarrow \infty} \int_{L_h^M} |\mathcal{H}(z) r^{-z}| |dz| &= \lim_{M \rightarrow \infty} \int_\ell^{-M} |\mathcal{H}(t + iM) r^{-t-iM}| dt \\ &\lesssim \lim_{M \rightarrow \infty} \int_\ell^{-M} e^{-\omega t} \left( \frac{\eta}{r} \right)^t M^{\omega t + \Lambda} \exp \left\{ -\frac{\alpha^* M \pi}{2} \right\} dt \\ &= \lim_{M \rightarrow \infty} \exp \left\{ -\frac{\alpha^* M \pi}{2} \right\} M^\Lambda \int_\ell^{-M} \left( \frac{\eta M^\omega}{r e^\omega} \right)^t dt \\ &= \lim_{M \rightarrow \infty} \exp \left\{ -\frac{\alpha^* M \pi}{2} \right\} M^\Lambda \cdot \frac{\left( \frac{\eta M^\omega}{r e^\omega} \right)^{-M} - \left( \frac{\eta M^\omega}{r e^\omega} \right)^\ell}{\ln \eta - \ln r + \omega(\ln M - 1)} = 0 \end{aligned}$$

since  $\alpha^* > 0$ ,  $\omega \geq 0$ , and  $r < \eta$ . Similarly,

$$\lim_{M \rightarrow \infty} \int_{L_h^{-M}} |\mathcal{H}(z) r^{-z}| |dz| = 0.$$

Thus, by taking  $p \rightarrow \infty$  in (3.25),

$$\begin{aligned} \sum_{k=0}^{\infty} \text{Res}_{z=\delta_k} [\mathcal{H}(z) r^{-z}] &= \lim_{M \rightarrow \infty} \frac{1}{2\pi i} \int_{C^M} \mathcal{H}(z) r^{-z} dz \\ &= \lim_{M \rightarrow \infty} \frac{1}{2\pi i} \left( \int_{L_v^M} + \int_{L_v^{-M}} + \int_{L_h^M} + \int_{L_h^{-M}} \right) \\ &= \lim_{M \rightarrow \infty} \frac{1}{2\pi i} \int_{L_v^M} \mathcal{H}(z) r^{-z} dz = \frac{1}{2\pi i} \int_{L_{Br}} \mathcal{H}(z) r^{-z} dz. \end{aligned}$$

Therefore (3.24) holds, and the proposition is proved.  $\square$

**Proposition 3.3** ([14, (2.2.2)]).

$$\frac{d}{dr} \left\{ H_{\nu\mu}^{mn} \left[ r \left| \begin{array}{c} [\mathfrak{c}, \gamma] \\ [\mathfrak{d}, \delta] \end{array} \right. \right] \right\} = -r^{-1} H_{\nu+1\mu+1}^{m+1\ n} \left[ r \left| \begin{array}{cc} [\mathfrak{c}, \gamma] & (0, 1) \\ (1, 1) & [\mathfrak{d}, \delta] \end{array} \right. \right].$$

*Proof.* Observe that  $\omega$  and  $\alpha^*$  of  $H_{\nu\mu}^{mn}(r)$  and of

$$H_{\nu+1\mu+1}^{m+1\ n} \left[ r \left| \begin{array}{cc} [\mathfrak{c}, \gamma] & (0, 1) \\ (1, 1) & [\mathfrak{d}, \delta] \end{array} \right. \right]$$

are same, respectively. Hence  $H_{\nu+1\mu+1}^{m+1\ n}(r)$  is well-defined with the same contour used for  $H_{\nu\mu}^{mn}(r)$ . Therefore the proposition easily follows from (3.1), (3.7)-(3.9), and the definition of the Fox H function.  $\square$

**3.2. Algebraic asymptotic expansions of  $H_{\nu\mu}^{mn}(r)$  near zero and at infinity.** Proposition 3.1 gives explicit power and power-logarithmic expansion of  $H_{\nu\mu}^{mn}(r)$  near zero for  $\omega \geq 0$  and at infinity for  $\omega \leq 0$ . For the reverse case (i.e.) near zero for  $\omega < 0$  and at infinity for  $\omega > 0$ , the following asymptotic expansions hold.

**Proposition 3.4.** *Suppose (3.21) holds. Then for sufficiently large  $p \in \mathbb{Z}_+$ , if  $\omega > 0$*

$$H_{\nu\mu}^{mn}(r) = \sum_{k=0}^p \text{Res}_{z=\hat{\mathfrak{c}}_k} [\mathcal{H}(r)r^{-z}] + O(r^{-M}), \quad \hat{\mathfrak{c}}_p < M < \hat{\mathfrak{c}}_{p+1} \quad (3.26)$$

as  $r \rightarrow \infty$ , and if  $\omega < 0$

$$H_{\nu\mu}^{mn}(r) = - \sum_{k=0}^p \text{Res}_{z=\hat{\mathfrak{d}}_k} [\mathcal{H}(r)r^{-z}] + O(r^M), \quad -\hat{\mathfrak{d}}_p < M < -\hat{\mathfrak{d}}_{p+1} \quad (3.27)$$

as  $r \rightarrow 0$ .

*Proof.* Braaksma [4] proved (3.26) for  $\omega \geq 0$ . We follow Braaksma's method to prove (3.27) for the case  $\omega < 0$ .

Let  $\omega < 0$ . Given (sufficiently large)  $p \in \mathbb{Z}_+$ , take a constant  $M > 0$  satisfying (3.14). Fix  $h > 0$  and let  $L_{Ha}^{-M}$  be a right Hankel contour surrounding  $L_{Ha}^+$ . Precisely,  $L_{Ha}^{-M}$  is a right loop situated in a horizontal strip runs from  $\infty - ih$  to  $-M - ih$  and then to  $-M + ih$  and finally terminating at the point  $\infty + ih$ . Let us denote by  $C$  a closed rectangular contour which encircles  $\hat{\mathfrak{d}}_0, \dots, \hat{\mathfrak{d}}_p$  and satisfies

$$\int_{L_{Ha}^+} \mathcal{H}(z)r^{-z}dz + \int_C \mathcal{H}(z)r^{-z}dz = \int_{L_{Ha}^{-M}} \mathcal{H}(z)r^{-z}dz.$$

Then

$$\begin{aligned} H_{\nu\mu}^{mn}(r) &= \frac{1}{2\pi i} \int_{L_{Ha}^+} \mathcal{H}(z)r^{-z}dz \\ &= \frac{1}{2\pi i} \left( - \int_C \mathcal{H}(z)r^{-z}dz + \int_{L_{Ha}^{-M}} \mathcal{H}(z)r^{-z}dz \right) \\ &= - \sum_{k=0}^p \text{Res}_{z=\hat{\mathfrak{d}}_k} [\mathcal{H}(z)r^{-z}] + \frac{1}{2\pi i} \int_{L_{Ha}^{-M}} \mathcal{H}(z)r^{-z}dz. \end{aligned} \quad (3.28)$$

Using (3.7), (3.8), and (3.21), and modifying the proof of Proposition 3.2,

$$\begin{aligned} \frac{1}{2\pi i} \int_{L_{Ha}^{-M}} \mathcal{H}(z)r^{-z}dz &= \sum_{k=0}^p \text{Res}_{z=\hat{\mathfrak{d}}_k} [\mathcal{H}(z)r^{-z}] + \sum_{k=0}^{\infty} \text{Res}_{z=\hat{\mathfrak{c}}_k} [\mathcal{H}(z)r^{-z}] \\ &= \frac{1}{2\pi i} \int_{L_{Br}^{-M}} \mathcal{H}(z)r^{-z}dz, \end{aligned}$$

where  $L_{Br}^{-M}$  is a Bromwich contour starting at the point  $-M - i\infty$  and terminating at the point  $-M + i\infty$ .

We estimate the upper bound of contour integral along  $L_{Br}^{-M}$ . Recall (3.9):

$$|\mathcal{H}(-M + it)r^{M-it}| \sim e^{\omega M} \eta^{-M} r^M |t|^{-\omega M + \Lambda} \exp \left\{ -\frac{|t|\alpha^*}{2} \pi \right\}$$

as  $t \rightarrow \infty$ . Thus, for  $r \leq 1$ ,

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{L_{Br}^{-M}} \mathcal{H}(z) r^{-z} dz \right| &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{H}(-M + it) r^{M-it}| dt \\ &\lesssim r^M \left( 1 + \frac{e^{\omega M}}{\eta^M} \int_{|t| \geq 1} |t|^{-\omega M + \Lambda} \exp \left\{ -\frac{|t|\alpha^*}{2} \pi \right\} dt \right) \\ &\lesssim r^M. \end{aligned}$$

The proposition is proved.  $\square$

If  $\omega > 0$  and  $n = 0$  (i.e.)  $P_2 = \emptyset$ , we have the following exponentially asymptotic behavior of  $H_{\nu\mu}^{m0}(r)$  (see [14, (1.7.13)]). For the proof we refer the reader to [4, Theorem 4].

**Proposition 3.5.** *Assume (3.21) and  $\omega > 0$ . Then*

$$H_{\nu\mu}^{m0}(r) = O \left( r^{(\Lambda + \frac{1}{2})/\omega} \exp \left\{ \cos \left( \frac{\alpha^* + \sum_{j=m+1}^{\mu} \delta_j}{\omega} \pi \right) \omega \left( \frac{r}{\eta} \right)^{1/\omega} \right\} \right)$$

as  $r \rightarrow \infty$ .

#### 4. ASYMPTOTIC ESTIMATES OF THE FOX H FUNCTION

Throughout this section we fix

$$d \in \mathbb{N}, \quad \alpha \in (0, 2), \quad \beta \in (0, \infty).$$

For  $\gamma \in [0, \infty)$ ,  $\sigma \in \mathbb{R}$  and  $z \in \mathbb{C}$  define

$$\mathcal{H}_{\sigma, \gamma}(z) := \frac{\Gamma(\frac{d}{2} + \gamma + \beta z) \Gamma(1 + z) \Gamma(-z)}{\Gamma(-\gamma - \beta z) \Gamma(1 - \sigma + \alpha z)}.$$

Observe that

$$\alpha^* = 2 - \alpha, \quad \Lambda = \frac{d}{2} + 2\gamma + \sigma - \frac{1}{2}, \quad \omega = 2\beta - \alpha, \quad \eta = \alpha^{-\alpha} \beta^{2\beta}.$$

For each  $k \in \mathbb{Z}_+$  we write

$$\mathfrak{c}_{1,k} := k, \quad \mathfrak{d}_{1,k} := -\frac{\frac{d}{2} + \gamma + k}{\beta}, \quad \mathfrak{d}_{2,k} := -1 - k.$$

Obviously, (3.5) holds, and thus  $\mathfrak{d}_{1,k}$ ,  $\mathfrak{d}_{2,k}$  and  $\mathfrak{c}_{1,k}$  constitute  $P_1$  and  $P_2$  respectively.

*Remark 4.1.* (i) Note that  $\mathcal{H}_{\sigma, \gamma}(z)$  has removable singularities at  $z = 0$  and at  $z = -1$  if  $\gamma = 0$  and  $\gamma = \beta$ , respectively. Indeed, by (3.1),

$$\mathcal{H}_{\sigma, 0}(z) = \frac{\beta \Gamma(\frac{d}{2} + \beta z) \Gamma(1 + z) \Gamma(1 - z)}{\Gamma(1 - \beta z) \Gamma(1 - \sigma + \alpha z)}.$$

Therefore,  $\mathcal{H}_{\sigma,0}(z)$  has a removable singularity at  $z = 0$ . Similarly,

$$\mathcal{H}_{\sigma,\beta}(z) = -\frac{\beta\Gamma(\frac{d}{2} + \beta + \beta z)\Gamma(2+z)\Gamma(-z)}{\Gamma(1-\beta-\beta z)\Gamma(1-\sigma+\alpha z)}.$$

Thus,  $\mathcal{H}_{\sigma,\beta}(z)$  has a removable singularity at  $z = -1$ .

(ii) Assume that  $\beta \in \mathbb{N}$  and  $\gamma = 0$ , then by (3.2),

$$\mathcal{H}_{\sigma,0}(z) = (2\pi)^{\frac{\beta-1}{2}} \frac{\Gamma(\frac{d}{2} + \beta z)\Gamma(1+z)}{\prod_{k=1}^{\beta-1} \Gamma(\frac{k}{\beta} - z)\Gamma(1-\sigma+\alpha z)} \beta^{\beta z + \frac{1}{2}}. \quad (4.1)$$

Hence  $P_2 = \emptyset$ .

(iii) If  $\alpha = 1$  and  $\sigma = 0$ , then

$$\mathcal{H}_{0,\gamma}(z) := \frac{\Gamma(\frac{d}{2} + \gamma + \beta z)\Gamma(-z)}{\Gamma(-\gamma - \beta z)}. \quad (4.2)$$

Thus,  $\mathfrak{d}_{2,k} = 0$  for all  $k \in \mathbb{Z}_+$ .

For  $r \in (0, \infty)$ , we define

$$\begin{aligned} \mathbb{H}_{\sigma,\gamma}(r) &:= \mathbb{H}_{23}^{21} \left[ r \middle| \begin{array}{cc} (1, 1) & (1-\sigma, \alpha) \\ (\frac{d}{2} + \gamma, \beta) & (1, 1) \end{array} \begin{array}{c} (1+\gamma, \beta) \end{array} \right] \\ &= \frac{1}{2\pi i} \int_L \mathcal{H}_{\sigma,\gamma}(z) r^{-z} dz. \end{aligned} \quad (4.3)$$

Here

$$L = L_{Br} = \{z \in \mathbb{C} : \Re[z] = \ell_0\}$$

and  $\ell_0$  is chosen to satisfy (3.11):

$$\max \left( -1, -\frac{\gamma}{\beta} - \frac{d}{2\beta} \right) < \ell_0 < 0$$

if  $\gamma \notin \{0, \beta\}$ ,

$$\max \left( -1, -\frac{d}{2\beta} \right) < \ell_0 < 1$$

if  $\gamma = 0$ ,

$$\max \left( -2, -1 - \frac{d}{2\beta} \right) < \ell_0 < 0$$

if  $\gamma = \beta$ . If  $\alpha = 1$  and  $\sigma = 0$ , then we take  $\ell_0$  such that

$$-\frac{\gamma}{\beta} - \frac{d}{2\beta} < \ell_0 < 0.$$

Since (3.21) holds (i.e.  $\alpha < 2$ ), by Propositions 3.1 and 3.2, the value of  $\mathbb{H}_{\sigma,\gamma}(r)$  is independent of the choice of  $\ell_0$  as long as it is chosen as above.

By Proposition 3.4, we obtain the asymptotic behaviors of  $\mathbb{H}_{\sigma,\gamma}(r)$  at infinity.

**Lemma 4.2.** *It holds that*

$$\mathbb{H}_{\sigma,\gamma}(r) = -\frac{\Gamma(\frac{d}{2} + \gamma)}{\Gamma(-\gamma)\Gamma(1-\sigma)} + O(r^{-1})$$

for  $r \geq 1$ . In particular, if  $\gamma \in \mathbb{Z}_+$  or  $\sigma \in \mathbb{N}$ , then

$$\mathbb{H}_{\sigma,\gamma}(r) = \frac{\Gamma(\frac{d}{2} + \gamma + \beta)}{\Gamma(-\gamma - \beta)\Gamma(1-\sigma+\alpha)} r^{-1} + O(r^{-2})$$



for  $r \geq 1$ . If  $\beta \in \mathbb{N}$  and  $\gamma = 0$ , then there exists a constant  $c = c(d, \alpha, \beta, \sigma) > 0$  such that

$$\mathbb{H}_{\sigma,0}(r) = O\left(\exp\left\{-cr^{1/(2\beta-\alpha)}\right\}\right)$$

as  $r \rightarrow \infty$ .

*Proof.* Observe that  $\mathcal{H}_{\sigma,\gamma}(z)$  has simple poles at  $z = \hat{\mathbf{c}}_k = \mathbf{c}_{1,k} = k$  for all  $k \in \mathbb{Z}_+$ . By (3.26), for sufficiently large  $p \in \mathbb{Z}_+$ ,

$$\begin{aligned} \mathbb{H}_{\sigma,\gamma}(r) &= \sum_{k=0}^1 \text{Res}_{z=\hat{\mathbf{c}}_k} [\mathcal{H}_{\sigma,\gamma}(z)r^{-z}] + \sum_{k=2}^p \text{Res}_{z=\hat{\mathbf{c}}_k} [\mathcal{H}_{\sigma,\gamma}(z)r^{-z}] + O(r^{-\hat{\mathbf{c}}_p}) \\ &= \sum_{k=0}^1 \text{Res}_{z=k} [\mathcal{H}_{\sigma,\gamma}(z)] r^{-k} + O(r^{-2}) \end{aligned}$$

for  $r \geq 1$ . Additionally, if  $\gamma \in \mathbb{Z}_+$  or  $\sigma \in \mathbb{N}$ , then

$$\begin{aligned} \text{Res}_{z=0} [\mathcal{H}_{\sigma,\gamma}(z)] &= \lim_{z \rightarrow 0} \left( \frac{z\Gamma(\frac{d}{2} + \gamma + \beta z)\Gamma(1+z)\Gamma(-z)}{\Gamma(-\gamma - \beta z)\Gamma(1 - \sigma + \alpha z)} \right) \\ &= -\lim_{z \rightarrow 0} \left( \frac{\Gamma(\frac{d}{2} + \gamma + \beta z)\Gamma(1+z)\Gamma(1-z)}{\Gamma(-\gamma - \beta z)\Gamma(1 - \sigma + \alpha z)} \right) = 0. \end{aligned}$$

Hence

$$\mathbb{H}_{\sigma,\gamma}(r) = \text{Res}_{z=1} [\mathcal{H}_{\sigma,\gamma}(z)] r^{-1} + O(r^{-2}) = \frac{\Gamma(\frac{d}{2} + \gamma + \beta)}{\Gamma(-\gamma - \beta)\Gamma(1 - \sigma + \alpha)} r^{-1} + O(r^{-2}).$$

Finally, assume  $\beta \in \mathbb{N}$  and  $\gamma = 0$ . By (4.1),

$$\begin{aligned} \mathbb{H}_{\sigma,0}(r) &= H_{23}^{21} \left[ r \begin{vmatrix} (1, 1) & (1 - \sigma, \alpha) \\ (\frac{d}{2}, \beta) & (1, 1) & (1, \beta) \end{vmatrix} \right] \\ &= \frac{(2\pi)^{\frac{\beta-1}{2}} \beta^{1/2}}{2\pi i} \int_L \frac{\Gamma(\frac{d}{2} + \beta z)\Gamma(1+z)}{\prod_{k=1}^{\beta-1} \Gamma(\frac{k}{\beta} - z)\Gamma(1 - \sigma + \alpha z)} \left( \frac{r}{\beta^\beta} \right)^{-z} dz \\ &= (2\pi)^{\frac{\beta-1}{2}} \beta^{1/2} H_{1\beta+1}^{20} \left[ \frac{r}{\beta^\beta} \begin{vmatrix} (1 - \sigma, \alpha) \\ (\frac{d}{2}, \beta) & (1, 1) & (1 + \frac{1}{\beta}, 1) & \dots & (1 + \frac{\beta-1}{\beta}, 1) \end{vmatrix} \right]. \end{aligned}$$

For the above Fox H function, we have

$$\alpha^* = 2 - \alpha, \quad \Lambda = \frac{d}{2} + \sigma + 2\beta - \frac{3}{2}, \quad \omega = 2\beta - \alpha, \quad \eta = \alpha^{-\alpha} \beta^\beta.$$

Note that

$$\frac{1}{2} < \frac{\beta + 1 - \alpha}{2\beta - \alpha} \leq 1, \quad 2\beta - \alpha \geq 2 - \alpha > 0.$$

Hence by Proposition 3.5,

$$\begin{aligned} &H_{1\beta+1}^{20} \left( \frac{r}{\beta^\beta} \right) \\ &= O \left( \left( \frac{r}{\beta^\beta} \right)^{(\Lambda + \frac{1}{2})/2\beta - \alpha} \exp \left\{ \cos \left( \frac{\beta + 1 - \alpha}{2\beta - \alpha} \pi \right) (2\beta - \alpha) \left( \frac{r}{\eta \beta^\beta} \right)^{1/(2\beta - \alpha)} \right\} \right) \\ &= O \left( \exp \left\{ -cr^{1/(2\beta - \alpha)} \right\} \right). \end{aligned}$$

The lemma is proved.  $\square$

Next we consider the asymptotic behavior of  $\mathbb{H}_{\sigma,\gamma}(r)$  at zero.

**Lemma 4.3.** *It holds that*

$$\mathbb{H}_{\sigma,\gamma}(r) \sim \begin{cases} r^{\frac{d+2\gamma}{2\beta}} & : \gamma < \beta - \frac{d}{2} \\ r|\ln r| & : \gamma = \beta - \frac{d}{2} \\ r & : \gamma > \beta - \frac{d}{2} \end{cases}$$

as  $r \rightarrow 0$ . Additionally, if  $\gamma - \beta \in \mathbb{Z}_+$  or  $\sigma + \alpha \in \mathbb{N}$ , then

$$\mathbb{H}_{\sigma,\gamma}(r) \sim \begin{cases} r^{\frac{d+2\gamma}{2\beta}} & : \gamma < 2\beta - \frac{d}{2} \\ r^2|\ln r| & : \gamma = 2\beta - \frac{d}{2} \\ r^2 & : \gamma > 2\beta - \frac{d}{2} \end{cases}$$

as  $r \rightarrow 0$ . If  $\alpha = 1$  and  $\sigma = 0$ , then

$$\mathbb{H}_{0,\gamma}(r) \sim r^{\frac{d+2\gamma}{2\beta}}$$

as  $r \rightarrow 0$ .

*Proof.* Due to (3.27), it is sufficient to compare the order of residues among  $z = \mathfrak{d}_{1,0}, \mathfrak{d}_{1,1}, \mathfrak{d}_{2,0}$ , and  $\mathfrak{d}_{2,1}$ .

First, let  $\gamma \neq \beta - \frac{d}{2}$ . Then  $\mathcal{H}_{\sigma,\gamma}(z)$  has a simple pole at  $z = \max\{\mathfrak{d}_{1,0}, \mathfrak{d}_{2,0}\}$ . If  $\gamma > \beta - \frac{d}{2}$ , then  $\mathfrak{d}_{1,0} < \mathfrak{d}_{2,0}$  so by (3.27)

$$\mathbb{H}_{\sigma,\gamma}(r) = \sum_{k=0}^1 \text{Res}_{z=\mathfrak{d}_{2,k}} [\mathcal{H}_{\sigma,\gamma}(z)r^{-z}] + O(r^{-\mathfrak{d}_{2,1}}) \sim r$$

as  $r \rightarrow 0$ . Similarly, if  $\gamma < \beta - \frac{d}{2}$ , then

$$\mathbb{H}_{\sigma,\gamma}(r) = \text{Res}_{z=\mathfrak{d}_{1,0}} [\mathcal{H}_{\sigma,\gamma}(z)r^{-z}] + O(r^{-\mathfrak{d}_{1,0}}) \sim r^{\frac{d+2\gamma}{2\beta}}$$

as  $r \rightarrow 0$ .

Next, assume  $\gamma = \beta - \frac{d}{2}$  (i.e.  $\mathfrak{d}_{1,0} = \mathfrak{d}_{2,0}$ ). Then  $\mathcal{H}_{\sigma,\gamma}(z)$  has a pole of order 2 at  $z = \hat{\mathfrak{d}}_0 = \mathfrak{d}_{1,0} = \mathfrak{d}_{2,0}$  so that

$$\begin{aligned} \text{Res}_{z=\hat{\mathfrak{d}}_0} [\mathcal{H}_{\sigma,\gamma}(z)r^{-z}] &= \lim_{z \rightarrow -1} \frac{d}{dz} ((z+1)^2 \mathcal{H}_{\sigma,\gamma}(z)r^{-z}) \\ &= \left( \text{Res}_{z=-1} [\mathcal{H}_{\sigma,\gamma}(z)] + \frac{|\ln r|}{\Gamma(\frac{d}{2})\Gamma(1-\sigma-\alpha)} \right) r. \end{aligned}$$

Then by (3.27), we obtain the first desired result.

Now we assume  $\gamma - \beta \in \mathbb{Z}_+$  or  $\sigma + \alpha \in \mathbb{N}$ . Then one can easily see that  $\text{Res}_{z=\mathfrak{d}_{2,0}} [\mathcal{H}_{\sigma,\gamma}(z)r^{-z}] = 0$ . Hence it remains to compare the order of residues at  $z = \mathfrak{d}_{1,0}, \mathfrak{d}_{1,1}$ , and  $\mathfrak{d}_{2,1}$ . Following the same argument of the above, we obtain the additional result.

Finally, we assume  $\alpha = 1$  and  $\sigma = 0$ . Recall (4.2), and note  $\mathfrak{d}_{2,k} = 0$  for all  $k \in \mathbb{Z}_+$  which implies

$$\mathbb{H}_{0,\gamma}(r) = \sum_{k=0}^{\infty} \text{Res}_{z=\mathfrak{d}_{1,k}} [\mathcal{H}_{0,\gamma}(z)r^{-z}] \sim r^{\frac{d+2\gamma}{2\beta}}$$

as  $r \rightarrow 0$ . The lemma is proved.  $\square$

For each  $q \in \mathbb{Z}_+$  we define

$$\mathcal{H}_{\sigma,\gamma}^{(q)}(z) := \mathcal{H}_{\sigma,\gamma}(z) \left\{ \frac{\Gamma(1+z)}{\Gamma(z)} \right\}^q \quad (4.4)$$

and

$$\mathbb{H}_{\sigma,\gamma}^{(q)}(r) := \frac{1}{2\pi i} \int_L \mathcal{H}_{\sigma,\gamma}^{(q)}(z) r^{-z} dz. \quad (4.5)$$

Note that  $\mathbb{H}_{\sigma,\gamma}^{(0)}(r) = \mathbb{H}_{\sigma,\gamma}(r)$  and by Proposition 3.3,

$$\frac{d}{dr} \mathbb{H}_{\sigma,\gamma}^{(q)}(r) = \mathbb{H}_{\sigma,\gamma}^{(q+1)}(r)$$

holds and (4.5) is well-defined for each  $q \in \mathbb{Z}_+$ . By (3.1)

$$\mathcal{H}_{\sigma,\gamma}^{(q)}(z) = -\frac{\Gamma(\frac{d}{2} + \gamma + \beta z) \Gamma(1-z) \Gamma(1+z)}{\Gamma(-\gamma - \beta z) \Gamma(1-\sigma + \alpha z)} z^{q-1},$$

and thus  $\mathcal{H}_{\sigma,\gamma}^{(q)}(z)$  does not have a pole at  $z = 0$  if  $q \geq 1$ . Furthermore,  $\mathcal{H}_{\sigma,\gamma}^{(q)}$  has a pole at  $z = -k - 1$  of order at most 2 for each  $k \in \mathbb{Z}_+$ . Let us denote by

$$\mathfrak{c}_{1,k} := k + 1, \quad \mathfrak{d}_{1,k} := -\frac{\frac{d}{2} + \gamma + k}{\beta}, \quad \mathfrak{d}_{2,k} := -1 - k$$

the new elements of  $P_1$  and  $P_2$  for  $\mathbb{H}_{\sigma,\gamma}^{(q)}(r)$ . Then due to (3.26) again, we obtain an analogue of Lemma 4.2.

**Lemma 4.4.** *Let  $q \in \mathbb{N}$ . It holds that*

$$\mathbb{H}_{\sigma,\gamma}^{(q)}(r) \sim r^{-1}$$

as  $r \rightarrow \infty$ . Additionally, if  $\beta \in \mathbb{N}$  and  $\gamma = 0$ , then there exists a constant  $c = c(d, \alpha, \beta, \sigma, |q|)$  such that

$$\mathbb{H}_{\sigma,0}^{(q)}(r) = O\left(\exp\left\{-cr^{1/(2\beta-\alpha)}\right\}\right)$$

as  $r \rightarrow \infty$ .

*Proof.* The proof is similar to the one of Lemma 4.2. The only difference is  $\hat{\mathfrak{c}}_k = \mathfrak{c}_{1,k} = k + 1$  which implies

$$\text{Res}_{z=\hat{\mathfrak{c}}_k} \left[ \mathcal{H}_{\sigma,\gamma}^{(q)}(z) r^{-z} \right] = \frac{(-1)^k \cdot \Gamma(\frac{d}{2} + \gamma + \beta + \beta k)}{\Gamma(-\gamma - \beta - \beta k) \Gamma(1 - \sigma + \alpha + \alpha k)} (k + 1)^q r^{-k-1}$$

for each  $k \in \mathbb{Z}_+$ . The lemma is proved.  $\square$

Lastly, we present another result which is necessary to obtain the upper estimates of classical derivatives of  $p(t, x)$ . Recall  $\omega = 2\beta - \alpha$ . Let  $\kappa_1$ ,  $\kappa_2$ ,  $\hat{\kappa}_1$ , and  $\hat{\kappa}_2$  denote constants

$$\kappa_1 := \text{Res}_{z=\mathfrak{d}_{1,0}} [\mathcal{H}_{\sigma,\gamma}(z)], \quad \kappa_2 := \text{Res}_{z=\mathfrak{d}_{2,0}} [\mathcal{H}_{\sigma,\gamma}(z)]$$

$$\hat{\kappa}_1 := \lim_{z \rightarrow \mathfrak{d}_{1,0}} (z - \mathfrak{d}_{1,0})^2 \mathcal{H}_{\sigma,\gamma}(z), \quad \hat{\kappa}_2 := \lim_{z \rightarrow \mathfrak{d}_{2,0}} (z - \mathfrak{d}_{2,0})^2 \mathcal{H}_{\sigma,\gamma}(z)$$

which are independent of  $q \geq 1$ . Note that  $\hat{\kappa}_2 = 0$  if  $\sigma + \alpha \in \mathbb{N}$ .

**Lemma 4.5.** *Let  $q \in \mathbb{Z}_+$  and  $\gamma \in [0, \infty)$ .*

*(i) If  $\gamma \notin \{\beta - \frac{d}{2}, \beta - \frac{d}{2} - 1, 2\beta - \frac{d}{2}\}$ , then*

$$\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r^{\min(1, \frac{d+2\gamma+2}{2\beta})}), \quad \left(\frac{0}{0} := 1\right)$$

*as  $r \rightarrow 0$ . Additionally, if  $\gamma = \beta$  or  $\sigma + \alpha \in \mathbb{N}$ , then*

$$\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + \begin{cases} O(r^{\min(2, \frac{d+2\gamma+2}{2\beta})}) & : \gamma \neq 2\beta - \frac{d}{2} - 1 \\ O(r^2 |\ln r|) & : \gamma = 2\beta - \frac{d}{2} - 1 \end{cases}$$

*as  $r \rightarrow 0$ . If  $\alpha = 1$  and  $\sigma = 0$ , then*

$$\mathbb{H}_{0, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r^{\frac{d+2\gamma+2}{2\beta}})$$

*as  $r \rightarrow 0$ .*

*(ii) If  $\gamma = \beta - \frac{d}{2}$ , then*

$$\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot (\hat{\kappa}_2 \ln r + \kappa_2) r + O(r)$$

*as  $r \rightarrow 0$ . Additionally, if  $\sigma + \alpha \in \mathbb{N}$ , then*

$$\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = -\frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_2 r + \begin{cases} O(r^{\min(2, \frac{d+2\gamma+2}{2\beta})}) & : \beta \neq 1 \\ O(r^2 |\ln r|) & : \beta = 1 \end{cases}$$

*as  $r \rightarrow 0$ .*

*(iii) If  $\gamma = \beta - \frac{d}{2} - 1$ , then*

$$\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r |\ln r|)$$

*as  $r \rightarrow 0$ . Additionally, if  $\sigma + \alpha \in \mathbb{N}$ , then*

$$\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r)$$

*as  $r \rightarrow 0$ .*

*(iv) If  $\gamma = 2\beta - \frac{d}{2}$ , then*

$$\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot (\hat{\kappa}_1 \ln r + \kappa_1) r^2 + O(r)$$

*as  $r \rightarrow 0$ . Additionally, if  $\sigma + \alpha \in \mathbb{N}$ , then*

$$\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot (\hat{\kappa}_1 \ln r + \kappa_1) r^2 + O(r^2)$$

*as  $r \rightarrow 0$ .*

*Proof.* Due to Proposition 3.1, one can easily see our assertions hold if  $\omega \geq 0$ . Hence, we assume  $\omega < 0$ . Recall (3.27) and take  $M \geq 2$  satisfying (3.14).

(i) If  $\gamma \neq \beta - \frac{d}{2}$ ,  $\gamma \neq \beta - \frac{d}{2} - 1$ , and  $\gamma \neq 2\beta - \frac{d}{2}$ , then  $\mathfrak{d}_{1,0} \neq \mathfrak{d}_{2,0}$ ,  $\mathfrak{d}_{1,1} \neq \mathfrak{d}_{2,0}$ , and  $\mathfrak{d}_{1,0} \neq \mathfrak{d}_{2,1}$ . Hence  $\mathcal{H}_{\sigma, \gamma}^{(q)}(z)$  has simple poles at  $\mathfrak{d}_{1,0} = -\frac{d+2\gamma}{2\beta}$  and  $\mathfrak{d}_{2,0} = -1$ .

Therefore,

$$\begin{aligned}
\mathbb{H}_{\sigma,\gamma}^{(q)}(r) &= - \sum_{j=1}^2 \operatorname{Res}_{z=\mathfrak{d}_{j,0}} \left[ \mathcal{H}_{\sigma,\gamma}^{(q)}(z) r^{-z} \right] - \operatorname{Res}_{z=\mathfrak{d}_{2,1}} \left[ \mathcal{H}_{\sigma,\gamma}^{(q)}(z) r^{-z} \right] + O(r^M) \\
&= - \left( \kappa_1 (\mathfrak{d}_{1,0})^q r^{-\mathfrak{d}_{1,0}} + \kappa_2 (\mathfrak{d}_{2,0})^q r^{-\mathfrak{d}_{2,0}} \right) \\
&\quad - \operatorname{Res}_{z=-2} \left[ \mathcal{H}_{\sigma,\gamma}^{(q)}(z) r^{-z} \right] + O(r^{\min(2, \frac{d+2\gamma+2}{2\beta})}) \\
&= - \left( -\frac{d+2\gamma}{2\beta} \right)^q \kappa_1 r^{\frac{d+2\gamma}{2\beta}} - (-1)^q \cdot \kappa_2 r \\
&\quad - \operatorname{Res}_{z=-2} \left[ \mathcal{H}_{\sigma,\gamma}^{(q)}(z) r^{-z} \right] + O(r^{\min(2, \frac{d+2\gamma+2}{2\beta})})
\end{aligned} \tag{4.6}$$

as  $r \rightarrow 0$ . Note that  $\kappa_2 = 0$  if  $\gamma = \beta$  or  $\sigma + \alpha \in \mathbb{N}$  so the second term of (4.6) vanishes. In that case, observe that

$$\operatorname{Res}_{z=-2} [\mathcal{H}_{\sigma,\gamma}^{(q)}(z) r^{-z}] = \begin{cases} O(r^2) & : \gamma \neq 2\beta - \frac{d}{2} - 1 \\ O(r^2 |\ln r|) & : \gamma = 2\beta - \frac{d}{2} - 1. \end{cases}$$

Furthermore, if  $\alpha = 1$  and  $\sigma = 0$ , then

$$\operatorname{Res}_{z=\mathfrak{d}_{2,k}} [\mathcal{H}_{\sigma,\gamma}^{(q)}(z) r^{-z}] = 0$$

for all  $k \in \mathbb{Z}_+$ . Thus (i) is proved.

(ii) Suppose  $\gamma = \beta - \frac{d}{2}$ . Then  $\hat{\mathfrak{d}}_0 = \mathfrak{d}_{1,0} = \mathfrak{d}_{2,0}$ ,  $\kappa_1 = \kappa_2$ , and  $\hat{\kappa}_1 = \hat{\kappa}_2$ . Note that  $\mathcal{H}_{\sigma,\gamma}^{(q)}(z)$  has a pole at

$$\hat{\mathfrak{d}}_0 = -\frac{d+2\gamma}{2\beta} = -1$$

of order 2. By (4.4),

$$\begin{aligned}
\operatorname{Res}_{z=-1} [\mathcal{H}_{\sigma,\gamma}^{(q)}(z) r^{-z}] &= \lim_{z \rightarrow -1} \frac{d}{dz} ((z+1)^2 \mathcal{H}_{\sigma,\gamma}(z) z^q r^{-z}) \\
&= (-1)^q \cdot \hat{\kappa}_2 r \ln r + \operatorname{Res}_{z=-1} [\mathcal{H}_{\sigma,\gamma}^{(q)}(z)] r \\
&= (-1)^q \cdot \hat{\kappa}_2 r \ln r + (-1)^q \cdot (\kappa_2 - q\hat{\kappa}_2) r \\
&= (-1)^q \cdot (\hat{\kappa}_2 \ln r + \kappa_2) r - q(-1)^q \hat{\kappa}_2 r
\end{aligned} \tag{4.7}$$

Therefore,

$$\mathbb{H}_{\sigma,\gamma}^{(q)}(r) = -(-1)^q \cdot (\hat{\kappa}_2 \ln r + \kappa_2) r + q(-1)^q \hat{\kappa}_2 r + \begin{cases} O(r^{\min(2, \frac{d+2\gamma+2}{2\beta})}) & : \beta \neq 1 \\ O(r^2 |\ln r|) & : \beta = 1 \end{cases}$$

as  $r \rightarrow 0$ . If we additionally assume  $\sigma + \alpha \in \mathbb{N}$ , then  $\hat{\kappa}_2 = 0$ . Thus (ii) is proved.

(iii) Now we let  $\gamma = \beta - \frac{d}{2} - 1$ . Then  $\mathfrak{d}_{1,1} = \mathfrak{d}_{2,0}$  and  $\mathcal{H}_{\sigma,\gamma}^{(q)}(z)$  has a pole at

$$-\frac{d+2\gamma+2}{2\beta} = \mathfrak{d}_{1,1} = \mathfrak{d}_{2,0} = -1$$

of order 2, and (4.7) holds. Also note that  $\mathcal{H}_{\sigma,\gamma}^{(q)}(z)$  has a simple pole at

$$z = \mathfrak{d}_{1,0} = -\frac{d+2\gamma}{2\beta} = \frac{1}{\beta} - 1.$$

Therefore,

$$\begin{aligned}\mathbb{H}_{\sigma,\gamma}^{(q)}(r) &= -\text{Res}_{z=-\frac{d+2\gamma}{2\beta}} \left[ \mathcal{H}_{\sigma,\gamma}^{(q)}(z)r^{-z} \right] - \text{Res}_{z=-1} \left[ \mathcal{H}_{\sigma,\gamma}^{(q)}(z)r^{-z} \right] + O(r) \\ &= - \left( -\frac{d+2\gamma}{2\beta} \right)^q \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r|\ln r|)\end{aligned}$$

as  $r \rightarrow 0$ . If  $\sigma + \alpha \in \mathbb{N}$ , then  $\hat{\kappa}_2 = 0$  and

$$\mathbb{H}_{\sigma,\gamma}^{(q)}(r) = - \left( -\frac{d+2\gamma}{2\beta} \right)^q \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r)$$

Thus (iii) is proved.

(iv) Finally, we assume  $\gamma = 2\beta - \frac{d}{2}$ . Then  $\mathfrak{d}_{1,0} = \mathfrak{d}_{2,1}$  and  $\mathcal{H}_{\sigma,\gamma}^{(q)}(z)$  has a pole at

$$-\frac{d+2\gamma}{2\beta} = \mathfrak{d}_{1,0} = \mathfrak{d}_{2,1} = -2$$

of order 2. By (4.4),

$$\begin{aligned}\text{Res}_{z=-2} \left[ \mathcal{H}_{\sigma,\gamma}^{(q)}(z)r^{-z} \right] &= \lim_{z \rightarrow -2} \frac{d}{dz} \left( (z+2)^2 \mathcal{H}_{\sigma,\gamma}(z) z^q r^{-z} \right) \\ &= \left( -\frac{d+2\gamma}{2\beta} \right) \cdot \hat{\kappa}_1 r^2 \ln r + \lim_{z \rightarrow -2} \frac{d}{dz} \left( (z+2)^2 \mathcal{H}_{\sigma,\gamma}(z) z^q \right) r^2 \\ &= \left( -\frac{d+2\gamma}{2\beta} \right)^q \cdot (\hat{\kappa}_1 \ln r + \kappa_1) r^2 + q \left( -\frac{d+2\gamma}{2\beta} \right)^{q-1} \hat{\kappa}_1 r^2.\end{aligned}$$

Note that  $\mathcal{H}_{\sigma,\gamma}^{(q)}(z)$  has a simple pole at  $z = \mathfrak{d}_{2,0} = -1$ . Therefore,

$$\begin{aligned}\mathbb{H}_{\sigma,\gamma}^{(q)}(r) &= -\text{Res}_{z=-\frac{d+2\gamma}{2\beta}} \left[ \mathcal{H}_{\sigma,\gamma}^{(q)}(z)r^{-z} \right] - \text{Res}_{z=-1} \left[ \mathcal{H}_{\sigma,\gamma}^{(q)}(z)r^{-z} \right] + O(r^2) \\ &= - \left( -\frac{d+2\gamma}{2\beta} \right)^q \cdot (\hat{\kappa}_1 \ln r + \kappa_1) r^2 - \kappa_2 r + O(r^2) \\ &= - \left( -\frac{d+2\gamma}{2\beta} \right)^q \cdot (\hat{\kappa}_1 \ln r + \kappa_1) r^2 + O(r)\end{aligned} \tag{4.8}$$

as  $r \rightarrow 0$ . Additionally, if  $\sigma + \alpha \in \mathbb{N}$ , then  $\kappa_2 = 0$  in (4.8) hence we obtain the desired result. The lemma is proved.  $\square$

## 5. EXPLICIT REPRESENTATION OF $p(t, x)$ AND ITS DERIVATIVES

Take the function  $\mathbb{H}_{\sigma,\gamma}(r)$  from (4.3), and define

$$p_{\sigma,\gamma}(t, x) := \frac{2^{2\gamma}}{\pi^{d/2}} |x|^{-d-2\gamma} t^{-\sigma} \mathbb{H}_{\sigma,\gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right), \quad (t, x) \in (0, \infty) \times \mathbb{R}_0^d.$$

Write

$$p(t, x) := p_{0,0}(t, x) = \frac{|x|^{-d}}{\pi^{d/2}} \mathbb{H}_{0,0} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right). \tag{5.1}$$

Note that  $\mathbb{H}_{\sigma,\gamma}(r)$  is a bounded function of  $r \in (0, \infty)$ . As a corollary of Lemmas 4.2 and 4.3, we obtain the following upper estimates of  $p_{\sigma,\gamma}(t, x)$  for  $\gamma \in [0, \infty)$  and  $\sigma \in \mathbb{R}$ .

**Theorem 5.1.** *Let  $\alpha \in (0, 2)$ ,  $\beta \in (0, \infty)$ ,  $\gamma \in [0, \infty)$ , and  $\sigma \in \mathbb{R}$ . Then for  $|x|^{2\beta}t^{-\alpha} \geq 1$*

$$|p_{\sigma,\gamma}(t, x)| \lesssim |x|^{-d-2\gamma}t^{-\sigma}$$

and for  $|x|^{2\beta}t^{-\alpha} \leq 1$

$$|p_{\sigma,\gamma}(t, x)| \lesssim \begin{cases} t^{-\sigma-\frac{\alpha(d+2\gamma)}{2\beta}} & : \gamma < \beta - \frac{d}{2} \\ |x|^{-d-2\gamma+2\beta}t^{-\sigma-\alpha} (1 + |\ln(|x|^{2\beta}t^{-\alpha})|) & : \gamma = \beta - \frac{d}{2} \\ |x|^{-d-2\gamma+2\beta}t^{-\sigma-\alpha} & : \gamma > \beta - \frac{d}{2}. \end{cases}$$

Furthermore,

(i) If  $\sigma + \alpha \in \mathbb{N}$ , then for  $|x|^{2\beta}t^{-\alpha} \leq 1$

$$|p_{\sigma,\gamma}(t, x)| \lesssim \begin{cases} t^{-\sigma-\frac{\alpha(d+2\gamma)}{2\beta}} & : \gamma < 2\beta - \frac{d}{2} \\ |x|^{-d-2\gamma+4\beta}t^{-\sigma-2\alpha} (1 + |\ln(|x|^{2\beta}t^{-\alpha})|) & : \gamma = 2\beta - \frac{d}{2} \\ |x|^{-d-2\gamma+4\beta}t^{-\sigma-2\alpha} & : \gamma > 2\beta - \frac{d}{2}. \end{cases}$$

(ii) If  $\alpha = 1$  and  $\sigma = 0$ , then for  $|x|^{2\beta}t^{-\alpha} \leq 1$

$$|p_{0,\gamma}(t, x)| \lesssim t^{-\frac{d+2\gamma}{2\beta}}.$$

(iii) If  $\gamma \in \mathbb{Z}_+$ , then for  $|x|^{2\beta}t^{-\alpha} \geq 1$

$$|p_{\sigma,\gamma}(t, x)| \lesssim |x|^{-d-2\gamma-2\beta}t^{-\sigma+\alpha}.$$

(iv) If  $\beta \in \mathbb{N}$  and  $\gamma = 0$ , then there exists a constant  $c = c(\alpha, \beta, \sigma) > 0$  such that for  $|x|^{2\beta}t^{-\alpha} \geq 1$ ,

$$|p_{\sigma,0}(t, x)| \lesssim |x|^{-d}t^{-\sigma} \exp \left\{ -c(t^{-\alpha}|x|^{2\beta})^{\frac{1}{2\beta-\alpha}} \right\}.$$

(v) If  $\gamma = \beta$ , then for  $|x|^{2\beta}t^{-\alpha} \geq 1$

$$|p_{\sigma,\beta}(t, x)| \lesssim \begin{cases} |x|^{-d-4\beta}t^{-\sigma+\alpha} & : \sigma \in \mathbb{N} \\ |x|^{-d-2\beta}t^{-\sigma} & : \sigma \in \mathbb{R} \setminus \mathbb{N} \end{cases}$$

and for  $|x|^{2\beta}t^{-\alpha} \leq 1$

$$|p_{\sigma,\beta}(t, x)| \lesssim \begin{cases} t^{-\sigma-\alpha-\frac{\alpha d}{2\beta}} & : \frac{d}{2} < \beta \\ |x|^{-d+2\beta}t^{-\sigma-2\alpha} (1 + |\ln(|x|^{2\beta}t^{-\alpha})|) & : \frac{d}{2} = \beta \\ |x|^{-d+2\beta}t^{-\sigma-2\alpha} & : \frac{d}{2} > \beta. \end{cases}$$

*Remark 5.2.* (i) By Theorem 5.1,  $p_{\sigma,\gamma}(t, \cdot) \in L_1(\mathbb{R}^d)$  for all  $\gamma \in [0, \beta]$  and  $\sigma \in \mathbb{R}$ . Furthermore, if  $\alpha = 1$  and  $\sigma = 0$ , then  $p_{0,\gamma}(t, \cdot) \in L_1(\mathbb{R}^d)$  for all  $\gamma \in [0, \infty)$ .

(ii) Observe that

$$p_{\sigma,\gamma}(t, x) = t^{-\sigma-\frac{\alpha(d+2\gamma)}{2\beta}} p_{\sigma,\gamma}(1, t^{-\frac{\alpha}{2\beta}}x) \quad (5.2)$$

which implies

$$\int_0^T \int_{\mathbb{R}^d} |p_{\sigma,\gamma}(t, x)| dx dt = \int_0^T t^{-\sigma-\frac{\alpha\gamma}{\beta}} \left( \int_{\mathbb{R}^d} |p_{\sigma,\gamma}(1, x)| dx \right) dt < \infty$$

if  $\sigma + \frac{\alpha\gamma}{\beta} < 1$ . Thus, under this condition, one can consider Riemann-Liouville fractional integral and the Fourier-Laplace transform of  $p_{\sigma,\gamma}(t, x)$ . However we do not use such transforms in this article.

The following theorem handles the interchangeability of  $\Delta^\gamma$  and  $\mathbb{D}_t^\sigma$ .

**Theorem 5.3.** *Let  $\gamma \in [0, \infty)$ . For any  $\sigma \in \mathbb{R}$ ,  $m \in \mathbb{N}$ , and  $\eta \in (-\infty, 1)$ ,*

$$D_t^m p_{\sigma, \gamma}(t, x) = p_{\sigma+m, \gamma}(t, x),$$

and

$$\mathbb{D}_t^\sigma p_{\eta, \gamma}(t, x) = p_{\sigma+\eta, \gamma}(t, x).$$

*Proof.* First we show

$$\frac{\partial}{\partial t} p_{\sigma, \gamma}(t, x) = p_{\sigma+1, \gamma}(t, x) \quad (5.3)$$

for any  $\sigma \in \mathbb{R}$ . By (3.1),

$$\frac{\alpha z}{\Gamma(1 - \sigma + \alpha z)} = \frac{1}{\Gamma(-\sigma + \alpha z)} + \frac{\sigma}{\Gamma(1 - \sigma + \alpha z)}$$

which implies

$$\mathcal{H}_{\sigma, \gamma}(z) \alpha z = \mathcal{H}_{\sigma+1, \gamma}(z) + \sigma \mathcal{H}_{\sigma, \gamma}(z).$$

Then by Proposition 3.3,

$$\begin{aligned} \frac{\partial}{\partial t} \mathbb{H}_{\sigma, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) &= \frac{t^{-1}}{2\pi i} \int_L \mathcal{H}_{\sigma, \gamma}(z) \alpha z \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right)^{-z} dz \\ &= \frac{t^{-1}}{2\pi i} \int_L (\mathcal{H}_{\sigma+1, \gamma}(z) + \sigma \mathcal{H}_{\sigma, \gamma}(z)) \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right)^{-z} dz \\ &= t^{-1} \left[ \mathbb{H}_{\sigma+1, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) + \sigma \mathbb{H}_{\sigma, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t} p_{\sigma, \gamma}(t, x) &= \frac{2^{2\gamma}}{\pi^{d/2}} |x|^{-d-2\gamma} t^{-\sigma} \left( \frac{\partial}{\partial t} \mathbb{H}_{\sigma, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) - \sigma t^{-1} \mathbb{H}_{\sigma, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) \right) \\ &= \frac{2^{2\gamma}}{\pi^{d/2}} |x|^{-d-2\gamma} t^{-\sigma-1} \mathbb{H}_{\sigma+1, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) \\ &= p_{\sigma+1, \gamma}(t, x), \end{aligned}$$

and (5.3) is proved. Thus to complete the proof of the theorem, due to (2.2) and (5.3), it is sufficient to show that for  $\sigma < 0$  and  $\eta < 1$

$$I_t^{|\sigma|} p_{\eta, \gamma}(t, x) = p_{\eta+\sigma, \gamma}(t, x).$$

Take  $\ell_0 > -\frac{1-\eta}{\alpha}$  satisfying (3.11). For  $\Re[z] > -\frac{1-\eta}{\alpha}$ , it holds that

$$\frac{1}{\Gamma(-\sigma)} \int_0^t (t-s)^{-\sigma-1} s^{-\eta+\alpha z} ds = \frac{\Gamma(1-\eta+\alpha z)}{\Gamma(1-\sigma-\eta+\alpha z)} t^{-\sigma-\eta+\alpha z}.$$

Observe that

$$\mathcal{H}_{\eta, \gamma}(z) \frac{\Gamma(1-\eta+\alpha z)}{\Gamma(1-\sigma-\eta+\alpha z)} = \mathcal{H}_{\eta+\sigma, \gamma}(z).$$



By (3.9) and the Fubini theorem,

$$\begin{aligned}
I_t^{|\sigma|} p_{\eta,\gamma}(t, x) &= \int_0^t \frac{(t-s)^{-\sigma-1}}{\Gamma(-\sigma)} p_{\eta,\gamma}(s, x) ds \\
&= \frac{2^{2\gamma} \pi^{-\frac{d}{2}} |x|^{-d-2\gamma}}{2\pi i} \int_0^t \int_L \frac{(t-s)^{-\sigma-1}}{\Gamma(-\sigma)} \mathcal{H}_{\eta,\gamma}(z) s^{-\eta} \left( \frac{|x|^{2\beta} s^{-\alpha}}{2^{2\beta}} \right)^{-z} dz ds \\
&= \frac{2^{2\gamma} \pi^{-\frac{d}{2}} |x|^{-d-2\gamma}}{2\pi i} \int_L \mathcal{H}_{\eta,\gamma}(z) \left( \frac{|x|^{2\beta}}{2^{2\beta}} \right)^{-z} \left[ \int_0^t \frac{(t-s)^{-\sigma-1}}{\Gamma(-\sigma)} s^{-\eta+\alpha z} ds \right] dz \\
&= \frac{2^{2\gamma} \pi^{-\frac{d}{2}} |x|^{-d-2\gamma}}{2\pi i} t^{-\sigma} \int_L \mathcal{H}_{\eta+\sigma,\gamma}(z) \left( \frac{|x|^{2\beta} t^{-\alpha}}{2^{2\beta}} \right)^{-z} dz \\
&= 2^{2\gamma} \pi^{-\frac{d}{2}} |x|^{-d-2\gamma} t^{-\eta-\sigma} \mathbb{H}_{\eta+\sigma,\gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) \\
&= p_{\eta+\sigma,\gamma}(t, x).
\end{aligned}$$

The theorem is proved.  $\square$

*Remark 5.4.* Let  $0 < \varepsilon < T$ . Then by Theorems 5.1 and 5.3,  $p(t, x)$  and  $\frac{\partial p}{\partial t}(t, x)$  are integrable in  $x \in \mathbb{R}^d$  uniformly in  $t \in [\varepsilon, T]$ . Thus

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^d} e^{-i(x,\xi)} p(t, x) dx = \int_{\mathbb{R}^d} e^{-i(x,\xi)} \frac{\partial p}{\partial t}(t, x) dx.$$

To estimate the classical derivatives of  $p(t, x)$ , we need the following theorem.

**Theorem 5.5.** *Let  $\alpha, \beta, \gamma, \sigma$  be given as in Theorem 5.1 and  $n \in \mathbb{N}$ . Then for  $|x|^{2\beta} t^{-\alpha} \geq 1$*

$$|D_x^n p_{\sigma,\gamma}(t, x)| \lesssim \begin{cases} |x|^{-d-n} t^{-\sigma} \exp \left\{ -(|x|^{2\beta} t^{-\alpha})^{\frac{1}{2\beta-\alpha}} \right\} & : \beta \in \mathbb{N}, \gamma = 0 \\ |x|^{-d-2\gamma-2\beta-n} t^{-\sigma+\alpha} & : \gamma \in \mathbb{Z}_+ \text{ or } \sigma \in \mathbb{N} \\ |x|^{-d-2\gamma-n} t^{-\sigma} & : \text{otherwise,} \end{cases}$$

and for  $|x|^{2\beta} t^{-\alpha} \leq 1$

$$|D_x^n p_{\sigma,\gamma}(t, x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma-\frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma < \beta - \frac{d}{2} - 1 \\ |x|^{2-n} t^{-\sigma-\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \gamma = \beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma+2\beta-n} t^{-\sigma-\alpha} & : \gamma > \beta - \frac{d}{2} - 1. \end{cases}$$

Additionally, for  $|x|^{2\beta} t^{-\alpha} \leq 1$ , the followings hold:

(i) If  $\alpha = 1$  and  $\sigma = 0$ ,

$$|D_x^n p_{0,\gamma}(t, x)| \lesssim |x|^{2-n} t^{-\frac{d+2\gamma+2}{2\beta}}.$$

(ii) If  $\sigma + \alpha \in \mathbb{N}$ ,

$$|D_x^n p_{\sigma,\gamma}(t, x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma-\frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma < 2\beta - \frac{d}{2} - 1 \\ |x|^{2-n} t^{-\sigma-2\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \gamma = 2\beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma+4\beta-n} t^{-\sigma-2\alpha} & : \gamma > 2\beta - \frac{d}{2} - 1. \end{cases}$$

(iii) If  $\gamma = \beta$ ,

$$|D_x^n p_{\sigma, \beta}(t, x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma-\alpha-\frac{\alpha(d+2)}{2\beta}} & : \frac{d}{2} + 1 < \beta \\ |x|^{2-n} t^{-\sigma-2\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \frac{d}{2} + 1 = \beta \\ |x|^{-d+2\beta-n} t^{-\sigma-2\alpha} & : \frac{d}{2} + 1 > \beta. \end{cases}$$

*Proof.* Write  $R = R(t, x) := 2^{-2\beta} |x|^{2\beta} t^{-\alpha}$  and recall  $\mathbb{H}_{\sigma, \gamma}^{(q)}$  from (4.5). By Proposition 3.3 and (3.1),

$$D_{x^i} \mathbb{H}_{\sigma, \gamma}^{(q)}(R) = -2\beta \frac{x^i}{|x|^2} \mathbb{H}_{\sigma, \gamma}^{(q+1)}(R)$$

for each  $q \in \mathbb{Z}_+$  and  $i = 1, \dots, d$ . Hence

$$\begin{aligned} & |D_{x^i} p_{\sigma, \gamma}(t, x)| \\ &= \left| -\frac{2^{2\gamma}}{\pi^{d/2}} x^i |x|^{-d-2\gamma-2} t^{-\sigma} \left( (d+2\gamma) \mathbb{H}_{\sigma, \gamma}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(1)}(R) \right) \right| \\ &\leq C |x|^{-d-2\gamma-1} t^{-\sigma} \left| (d+2\gamma) \mathbb{H}_{\sigma, \gamma}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(1)}(R) \right| \end{aligned}$$

and

$$\begin{aligned} & |D_{x^j} D_{x^i} p_{\sigma, \gamma}(t, x)| \\ &= \left| D_{x^j} \left\{ -\frac{2^{2\gamma}}{\pi^{d/2}} x^i |x|^{-d-2\gamma-2} t^{-\sigma} \left( (d+2\gamma) \mathbb{H}_{\sigma, \gamma}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(1)}(R) \right) \right\} \right| \\ &= \frac{2^{2\gamma}}{\pi^{d/2}} t^{-\sigma} \left| -\delta_{ij} |x|^{-d-2\gamma-2} \left\{ (d+2\gamma) \mathbb{H}_{\sigma, \gamma}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(1)}(R) \right\} \right. \\ &\quad \left. + (d+2\gamma+2) x^i x^j |x|^{-d-2\gamma-4} \left\{ (d+2\gamma) \mathbb{H}_{\sigma, \gamma}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(1)}(R) \right\} \right. \\ &\quad \left. + 2\beta x^i x^j |x|^{-d-2\gamma-4} \left\{ (d+2\gamma) \mathbb{H}_{\sigma, \gamma}^{(1)}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(2)}(R) \right\} \right| \\ &\leq C |x|^{-d-2\gamma-2} t^{-\sigma} \sum_{q=1}^2 \left| (d+2\gamma) \mathbb{H}_{\sigma, \gamma}^{(q-1)}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(q)}(R) \right|. \end{aligned}$$

Inductively, for any  $n \in \mathbb{N}$ ,

$$|D_x^n p_{\sigma, \gamma}(t, x)| \leq C |x|^{-d-2\gamma-n} t^{-\sigma} \sum_{q=1}^n \left| (d+2\gamma) \mathbb{H}_{\sigma, \gamma}^{(q-1)}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(q)}(R) \right|. \quad (5.4)$$

By Lemma 4.2, 4.4, and (5.4),

$$|D_x^n p_{\sigma, \gamma}(t, x)| \lesssim \begin{cases} t^{\frac{\alpha(d+n)}{2\beta}-\sigma} \exp \left\{ -c(|x|^{2\beta} t^{-\alpha})^{\frac{1}{2\beta-\alpha}} \right\} & : \gamma = 0, \beta \in \mathbb{N} \\ |x|^{-d-2\gamma-2\beta-n} t^{-\sigma+\alpha} & : \gamma \in \mathbb{Z}_+ \text{ or } \sigma \in \mathbb{N} \\ |x|^{-d-2\gamma-n} t^{-\sigma} & : \text{otherwise} \end{cases}$$

for  $|x|^{2\beta} t^{-\alpha} \geq 1$ . To estimate the upper bounds for  $|x|^{2\beta} t^{-\alpha} \leq 1$ , observe that

$$(d+2\gamma) \left( -\frac{d+2\gamma}{2\beta} \right)^{q-1} + 2\beta \left( -\frac{d+2\gamma}{2\beta} \right)^q = 0.$$

Thus, by Lemma 4.5 and (5.4),

$$|D_x^n p_{\sigma,\gamma}(t, x)| \lesssim \begin{cases} |x|^{2-|n|} t^{-\sigma-\frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma < \beta - \frac{d}{2} - 1 \\ |x|^{2-|n|} t^{-\sigma-\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \gamma = \beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma+2\beta-|n|} t^{-\sigma-\alpha} & : \gamma > \beta - \frac{d}{2} - 1 \end{cases}$$

for  $R \leq 1$ .

Additionally, if  $\sigma + \alpha \in \mathbb{N}$ , by Lemma 4.5,

$$|D_x^n p_{\sigma,\gamma}(t, x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma-\frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma < 2\beta - \frac{d}{2} - 1 \\ |x|^{2-|n|} t^{-\sigma-\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \gamma = 2\beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma+4\beta-|n|} t^{-\sigma-2\alpha} & : \gamma > 2\beta - \frac{d}{2} - 1 \end{cases}$$

for  $R \leq 1$ . Also, if  $\gamma = \beta$ ,

$$|D_x^n p_{\sigma,\beta}(t, x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma-\alpha-\frac{\alpha(d+2)}{2\beta}} & : \frac{d}{2} + 1 < \beta \\ |x|^{2-|n|} t^{-\sigma-2\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \frac{d}{2} + 1 = \beta \\ |x|^{-d+2\beta-|n|} t^{-\sigma-2\alpha} & : \frac{d}{2} + 1 > \beta \end{cases}$$

for  $R \leq 1$ . The theorem is proved.  $\square$

## 6. PROOFS OF THEOREMS 2.1, 2.3 AND 2.4

By Remark 5.2,  $p_{\sigma,\gamma}(t, \cdot) \in L_1(\mathbb{R}^d)$  if  $\gamma \in [0, \beta]$  and  $\sigma \in \mathbb{R}$ , and  $p_{0,\gamma}(t, \cdot) \in L_1(\mathbb{R}^d)$  if  $\alpha = 1$ ,  $\gamma \in [0, \infty)$ , and  $\sigma = 0$ . Due to Lemmas 4.2, 4.3, Theorems 5.1, 5.3, and 5.5, to prove our desired results, it is enough to show

$$\mathcal{F}\{p_{\sigma,\gamma}(t, \cdot)\} = |\xi|^{2\gamma} t^{-\sigma} E_{\alpha,1-\sigma}(-t^\alpha |\xi|^{2\beta}), \quad (6.1)$$

which is equivalent to

$$p_{\sigma,\gamma}(t, x) = \Delta^\gamma p_{\sigma,0}(t, x) = \Delta^\gamma \mathbb{D}_t^\sigma p(t, x).$$

We devide the proof into the cases  $d \geq 2$  and  $d = 1$ .

**Case 1:**  $d \geq 2$ . We choose  $\ell_0 \in \mathbb{R}$  in (4.3) such that

$$\max(-2, -1 - \frac{d-1}{4\beta}) < \ell_0 < -1 \quad (6.2)$$

if  $\gamma = \beta$ ,

$$\max(-1, -\frac{\gamma}{\beta} - \frac{d-1}{4\beta}) < \ell_0 < -\frac{\gamma}{\beta} \quad (6.3)$$

if  $\gamma \in (0, \beta)$ ,

$$\max(-1, -\frac{1}{\alpha}, -\frac{d-1}{4\beta}) < \ell_0 < 0 \quad (6.4)$$

if  $\gamma = 0$ . If  $\alpha = 1$ ,  $\gamma \neq 0$ , and  $\sigma = 0$ , then we take  $\ell_0$  such that

$$-\frac{\gamma}{\beta} - \frac{d-1}{4\beta} < \ell_0 < -\frac{\gamma}{\beta}. \quad (6.5)$$

Under the above restriction on  $\ell_0$ , (4.3) is well-defined and the value of  $p_{\sigma,\gamma}(t, x)$  is independent of the choice of  $\ell_0$ .

Due to the Fourier transform for radial function (see [34, Theorem IV.3.3])

$$\begin{aligned} & \mathcal{F}\{p_{\sigma,\gamma}(t, \cdot)\}(\xi) \\ &= \frac{2^{\frac{d}{2}+2\gamma}}{|\xi|^{\frac{d}{2}-1}} t^{-\sigma} \int_0^\infty \rho^{-\frac{d}{2}-2\gamma} \mathbb{H}_{\sigma,\gamma}\left(\frac{\rho^{2\beta}}{2^{2\beta}} t^{-\alpha}\right) J_{\frac{d}{2}-1}(|\xi|\rho) d\rho \end{aligned}$$

where  $J_{\frac{d}{2}-1}$  is the Bessel function of the first kind of order  $\frac{d}{2}-1$ , (i.e.) for  $r \in [0, \infty)$ ,

$$J_{\frac{d}{2}-1}(r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \frac{d}{2})} \left(\frac{r}{2}\right)^{2k-1+\frac{d}{2}}.$$

It is well-known if  $m > -1$  then

$$J_m(t) = \begin{cases} O(t^m) & : t \rightarrow 0+ \\ O(t^{-1/2}) & : t \rightarrow \infty. \end{cases}$$

Due to (6.2)-(6.5),

$$\int_0^\infty \left| \rho^{-\frac{d}{2}-2\gamma-2\beta\ell_0} J_{\frac{d}{2}-1}(|\xi|\rho) \right| d\rho \leq \int_0^1 \rho^{-2\beta\ell_0-2\gamma-1} d\rho + \int_1^\infty \rho^{-\frac{d}{2}-2\gamma-2\beta\ell_0-\frac{1}{2}} d\rho < \infty.$$

Recall  $\Re[z] = \ell_0$  along  $L$ . By the above and (3.9),

$$\int_0^\infty \int_L \left| \rho^{-\frac{d}{2}} \mathcal{H}_{\sigma,\gamma}(z) \left( \frac{\rho^{2\beta} t^{-\alpha}}{2^{2\beta}} \right)^{-z} J_{\frac{d}{2}-1}(|\xi|\rho) \right| |dz| d\rho < \infty.$$

Therefore, by the Fubini theorem

$$\begin{aligned} & \int_0^\infty \rho^{-\frac{d}{2}-2\gamma} \mathbb{H}_{\sigma,\gamma}\left(\frac{\rho^{2\beta}}{2^{2\beta}} t^{-\alpha}\right) J_{\frac{d}{2}-1}(|\xi|\rho) d\rho \\ &= \frac{1}{2\pi i} \int_0^\infty \rho^{-\frac{d}{2}-2\gamma} \left[ \int_L \mathcal{H}_{\sigma,\gamma}(z) \left( \frac{\rho^{2\beta} t^{-\alpha}}{2^{2\beta}} \right)^{-z} dz \right] J_{\frac{d}{2}-1}(|\xi|\rho) d\rho \\ &= \frac{1}{2\pi i} \int_L \left[ \int_0^\infty \rho^{-\frac{d}{2}-2\gamma-2\beta z} J_{\frac{d}{2}-1}(|\xi|\rho) d\rho \right] \mathcal{H}_{\sigma,\gamma}(z) \left( \frac{t^{-\alpha}}{2^{2\beta}} \right)^{-z} dz. \end{aligned}$$

By using the formula [1, (11.4.16)],

$$\int_0^\infty \rho^{-\frac{d}{2}-2\gamma-2\beta z} J_{\frac{d}{2}-1}(|\xi|\rho) d\rho = 2^{-\frac{d}{2}-2\gamma-2\beta z} |\xi|^{\frac{d}{2}+2\gamma+2\beta z-1} \frac{\Gamma(-\gamma-\beta z)}{\Gamma(\frac{d}{2}+\gamma+\beta z)}$$

we have

$$\mathcal{H}_{\sigma,\gamma}(z) \frac{\Gamma(-\gamma-\beta z)}{\Gamma(\frac{d}{2}+\gamma+\beta z)} = \frac{\Gamma(z+1)\Gamma(-z)}{\Gamma(1-\sigma+\alpha z)}.$$

Hence

$$\begin{aligned}
& \frac{2^{\frac{d}{2}+2\gamma}}{|\xi|^{\frac{d}{2}-1}} t^{-\sigma} \cdot \frac{1}{2\pi i} \int_L \left[ \int_0^\infty \rho^{-\frac{d}{2}-2\gamma-2\beta z} J_{\frac{d}{2}-1}(|\xi|\rho) d\rho \right] \mathcal{H}_{\sigma,\gamma}(z) \left( \frac{t^{-\alpha}}{2^{2\beta}} \right)^{-z} dz \\
&= \frac{2^{\frac{d}{2}+2\gamma}}{|\xi|^{\frac{d}{2}-1}} t^{-\sigma} \cdot \frac{2^{-\frac{d}{2}-2\gamma}}{2\pi i} |\xi|^{\frac{d}{2}+2\gamma-1} \int_L \frac{\Gamma(z+1)\Gamma(-z)}{\Gamma(1-\sigma+\alpha z)} (|\xi|^{-2\beta} t^{-\alpha})^{-z} dz \\
&= |\xi|^{2\gamma} t^{-\sigma} \cdot \frac{1}{2\pi i} \int_{-L} \frac{\Gamma(1-z)\Gamma(z)}{\Gamma(1-\sigma-\alpha z)} (|\xi|^{2\beta} t^\alpha)^{-z} dz \\
&= |\xi|^{2\gamma} t^{-\sigma} \mathbf{H}_{12}^{11} \left[ \begin{matrix} (0,1) \\ (0,1) \end{matrix} \middle| \begin{matrix} (0,1) \\ (\sigma,\alpha) \end{matrix} \right] \\
&= |\xi|^{2\gamma} t^{-\sigma} E_{\alpha,1-\sigma}(-|\xi|^{2\beta} t^\alpha).
\end{aligned}$$

Therefore,

$$\mathcal{F}\{p_{\sigma,\gamma}(t, \cdot)\} = |\xi|^{2\gamma} t^{-\sigma} E_{\alpha,1-\sigma}(-|\xi|^{2\beta} t^\alpha).$$

**Case 2:**  $d = 1, \gamma \in (0, \beta)$ . We choose  $\ell_0$  such that

$$\max(-1, -\frac{\gamma}{\beta} - \frac{1}{2\beta}) < \ell_0 < 0.$$

Since  $p_{\sigma,\gamma}(t, x)$  is an even function,

$$\begin{aligned}
\mathcal{F}\{p_{\sigma,\gamma}(t, \cdot)\} &= \int_{-\infty}^\infty e^{-i\xi x} p_{\sigma,\gamma}(t, x) dx \\
&= 2 \int_0^\infty p_{\sigma,\gamma}(t, x) \cos(\xi x) dx \\
&= 2 \left( \int_0^{t^{\alpha/2\beta}} p_{\sigma,\gamma}(t, x) \cos(\xi x) dx + \int_{t^{\alpha/2\beta}}^\infty p_{\sigma,\gamma}(t, x) \cos(\xi x) dx \right) \\
&= 2t^{-\sigma-\frac{\alpha\gamma}{\beta}} \left( \int_0^1 p_{\sigma,\gamma}(1, x) \cos(t^{\frac{\alpha}{2\beta}} \xi x) dx + \int_1^\infty p_{\sigma,\gamma}(1, x) \cos(t^{\frac{\alpha}{2\beta}} \xi x) dx \right).
\end{aligned}$$

The last equality holds due to (5.2). Set Bromwich contours

$$L_{<c} := \{z \in \mathbb{C} : \Re[z] = \ell_{<c}\}, \quad L_{>c} := \{z \in \mathbb{C} : \Re[z] = \ell_{>c}\}$$

where

$$\max(-1, -\frac{\gamma}{\beta} - \frac{1}{2\beta}) < \ell_{<c} < -\frac{\gamma}{\beta}, \quad -\frac{\gamma}{\beta} < \ell_{>c} < 0.$$

By (3.9),

$$\begin{aligned}
& \int_0^1 \int_{L_{<c}} |2^{2\beta z} \mathcal{H}_{\sigma,\gamma}(z) x^{-2\gamma-2\beta z-1}| |dz| dx \\
& \quad + \int_1^\infty \int_{L_{>c}} |2^{2\beta z} \mathcal{H}_{\sigma,\gamma}(z) x^{-2\gamma-2\beta z-1}| |dz| dx < \infty.
\end{aligned}$$

Therefore, by the Fubini theorem,

$$\begin{aligned}
& \int_0^1 p_{\sigma,\gamma}(1, x) \cos(t^{\frac{\alpha}{2\beta}} \xi x) dx \\
&= \frac{2^{2\gamma} \pi^{-1/2}}{2\pi i} \int_0^1 \int_{L_{<c}} 2^{2\beta z} \mathcal{H}_{\sigma,\gamma}(z) x^{-2\gamma-2\beta z-1} \cos(t^{\frac{\alpha}{2\beta}} \xi x) dz dx \\
&= \frac{2^{2\gamma} \pi^{-1/2}}{2\pi i} \int_{L_{<c}} 2^{2\beta z} \mathcal{H}_{\sigma,\gamma}(z) \left[ \int_0^1 x^{-2\gamma-2\beta z-1} \cos(t^{\frac{\alpha}{2\beta}} \xi x) dx \right] dz.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_1^\infty p_{\sigma,\gamma}(1, x) \cos(t^{\frac{\alpha}{2\beta}} \xi x) dx \\
&= \frac{2^{2\gamma} \pi^{-1/2}}{2\pi i} \int_{L_{>c}} 2^{2\beta z} \mathcal{H}_{\sigma,\gamma}(z) \left[ \int_1^\infty x^{-2\gamma-2\beta z-1} \cos(t^{\frac{\alpha}{2\beta}} \xi x) dx \right] dz.
\end{aligned}$$

For  $\eta \in \mathbb{R}$ , it holds that (see [30, (1.8.1.1), (1.8.1.2)])

$$\int_0^1 x^{-2\lambda-1} \cos(\eta x) dx = -\frac{{}_1F_2\left(-\lambda; \frac{1}{2}, 1-\lambda; -\frac{\eta^2}{4}\right)}{2\lambda}, \quad \Re[\lambda] < 0$$

and

$$\int_1^\infty x^{-2\lambda-1} \cos(\eta x) dx = \frac{\Gamma(-2\lambda) \cos \lambda \pi}{|\eta|^{-2\lambda}} + \frac{{}_1F_2\left(-\lambda; \frac{1}{2}, 1-\lambda; -\frac{\eta^2}{4}\right)}{2\lambda}, \quad \Re[\lambda] > -\frac{1}{2},$$

where  ${}_1F_2\left(-\lambda; \frac{1}{2}, 1-\lambda; -\frac{\eta^2}{4}\right)$  denotes the general hypergeometric function, i.e.

$${}_1F_2(p; q, r; z) = \frac{\Gamma(q)\Gamma(r)}{\Gamma(p)} \sum_{k=0}^\infty \frac{\Gamma(p+k)}{\Gamma(q+k)\Gamma(r+k)} \frac{z^k}{k!}, \quad p \in \mathbb{C}, \quad q, r \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

Observe that for  $\gamma + \beta z \in \mathbb{C} \setminus \mathbb{N}$ ,

$$\begin{aligned}
& {}_1F_2\left(-\gamma - \beta z; \frac{1}{2}, 1 - \gamma - \beta z; -\frac{t^{\frac{\alpha}{2\beta}} |\xi|^2}{4}\right) \\
&= \frac{\sqrt{\pi} \Gamma(1 - \gamma - \beta z)}{\Gamma(-\gamma - \beta z)} \sum_{k=0}^\infty \frac{\Gamma(-\gamma - \beta z + k)}{\Gamma(\frac{1}{2} + k) \Gamma(1 - \gamma - \beta z + k)} \frac{\left(-t^{\frac{\alpha}{2\beta}} |\xi|^2 / 4\right)^k}{k!} \\
&= -\sum_{k=0}^\infty \frac{\sqrt{\pi} (\gamma + \beta z)}{\Gamma(\frac{1}{2} + k) k! (-\gamma - \beta z + k)} \left(-\frac{t^{\frac{\alpha}{2\beta}} |\xi|^2}{4}\right)^k \\
&= 1 - \sum_{k=1}^\infty \frac{\gamma + \beta z}{(2k-1)! 2k (-\gamma - \beta z + k)} \left(-\frac{t^{\frac{\alpha}{2\beta}} |\xi|^2}{4}\right)^k =: F_{t,\xi}(z).
\end{aligned}$$

Since the series in  $F_{t,\xi}(z)$  converges absolutely for fixed  $t, \xi$ , one can easily see that  $F_{t,\xi}(z)$  is holomorphic in  $\{z \in \mathbb{C} : -\frac{2\gamma-1}{2\beta} < \Re[z] < -\frac{2\gamma+1}{2\beta}\}$  by Morera's theorem. So we obtain

$$\int_0^1 p_{\sigma,\gamma}(1, x) \cos(t^{\frac{\alpha}{2\beta}} \xi x) dx = -\frac{2^{2\gamma} \pi^{-1/2}}{2\pi i} \int_{L_{<c}} 2^{2\beta z} \frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma + \beta z)} F_{t,\xi}(z) dz,$$

and

$$\begin{aligned} & \int_1^\infty p_{\sigma,\gamma}(1, x) \cos(t^{\frac{\alpha}{2\beta}} \xi x) dx \\ &= \frac{2^{2\gamma} \pi^{-1/2}}{2\pi i} \left( \int_{L_{>c}} 2^{2\beta z} \mathcal{H}_{\sigma,\gamma}(z) \Gamma(-2\gamma - 2\beta z) \cos((\gamma + \beta z)\pi) \left| t^{\frac{\alpha}{2\beta}} \xi \right|^{2\gamma+2\beta z} dz \right. \\ & \quad \left. + \int_{L_{>c}} 2^{2\beta z} \frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma + \beta z)} F_{t,\xi}(z) dz \right). \end{aligned}$$

If  $\gamma \neq \beta$ , by (3.1),

$$\frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma + \beta z)} = -\frac{\Gamma(\frac{1}{2} + \gamma + \beta z) \Gamma(1 + z) \Gamma(-z)}{2\Gamma(1 - \gamma - \beta z) \Gamma(1 - \sigma + \alpha z)}.$$

Thus  $z = -\frac{\gamma}{\beta}$  is a removable singularity. These facts lead to

$$\text{Res}_{z=-\frac{\gamma}{\beta}} \left[ 2^{2\beta z} \frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma + \beta z)} F_{t,\xi}(z) \right] = 0 \quad (6.6)$$

and

$$\frac{1}{2\pi i} \int_{L_{>c}} 2^{2\beta z} \frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma + \beta z)} F_{t,\xi}(z) dz = \frac{1}{2\pi i} \int_{L_{<c}} 2^{2\beta z} \frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma + \beta z)} F_{t,\xi}(z) dz.$$

Therefore,

$$\begin{aligned} & \int_0^\infty p_{\sigma,\gamma}(1, x) \cos(t^{\frac{\alpha}{2\beta}} \xi x) dx \\ &= \frac{2^{2\gamma} \pi^{-1/2} t^{\frac{\alpha\gamma}{\beta}} |\xi|^{2\gamma}}{2\pi i} \int_{L_{>c}} 2^{2\beta z} \mathcal{H}_{\sigma,\gamma}(z) \Gamma(-2\gamma - 2\beta z) \cos((\gamma + \beta z)\pi) (t^\alpha |\xi|^{2\beta})^z dz. \end{aligned}$$

By (3.2),

$$\mathcal{H}_{\sigma,\gamma}(z) \Gamma(-2\gamma - 2\beta z) \cos((\gamma + \beta z)\pi) = \pi^{1/2} 2^{-2\gamma-2\beta z-1} \frac{\Gamma(1+z) \Gamma(-z)}{\Gamma(1-\sigma+\alpha z)}.$$

Hence

$$\begin{aligned} \mathcal{F} \{p_{\sigma,\gamma}(t, \cdot)\}(\xi) &= 2t^{-\sigma-\frac{\alpha\gamma}{\beta}} \int_0^\infty p_{\sigma,\gamma}(1, x) \cos(t^{\frac{\alpha}{2\beta}} \xi x) dx \\ &= \frac{|\xi|^{2\gamma} t^{-\sigma}}{2\pi i} \int_L \mathcal{H}_{\sigma,\gamma}(z) \frac{\Gamma(1+z) \Gamma(-z)}{\Gamma(1-\sigma+\alpha z)} (t^\alpha |\xi|^{2\beta})^z dz \\ &= |\xi|^{2\gamma} t^{-\sigma} E_{\alpha,1-\sigma}(-|\xi|^{2\beta} t^\alpha). \end{aligned}$$

**Case 3:**  $d = 1, \gamma = 0$ . Let

$$\max(-1, -\frac{1}{\alpha}, -\frac{1}{2\beta}) < \ell_0 < 1.$$

Again, we follow the argument in Case 2. Note that

$$\frac{\mathcal{H}_{\sigma,0}(z)}{2\beta z} = -\frac{\Gamma(\frac{1}{2} + \beta z) \Gamma(1 + z) \Gamma(-z)}{2\Gamma(1 - \beta z) \Gamma(1 - \sigma + \alpha z)}$$

has a removable singularity at  $z = 0$  if and only if  $\sigma \in \mathbb{N}$ . Thus (6.6) holds if  $\sigma = 1$  and we immediately obtain

$$\mathcal{F} \{p_{1,0}(t, \cdot)\} = t^{-1} E_{\alpha,0}(-|\xi|^{2\beta} t^\alpha) = \frac{\partial}{\partial t} E_\alpha(-|\xi|^{2\beta} t^\alpha).$$

By Remark 5.4 and Theorem 5.3,

$$\frac{\partial}{\partial t} \mathcal{F} \{p(t, \cdot)\} = \mathcal{F} \left\{ \frac{\partial p}{\partial t}(t, \cdot) \right\} = \mathcal{F} \{p_{1,0}(t, \cdot)\}.$$

Thus,

$$\mathcal{F} \{p(t, \cdot)\}(\xi) = E_\alpha(-|\xi|^{2\beta} t^\alpha) + R(\xi).$$

By (5.2),

$$\begin{aligned} \mathcal{F} \{p(t, \cdot)\}(\xi) &= \int_{\mathbb{R}} e^{-ix\xi} p(t, x) dx = \int_{\mathbb{R}} e^{-ix\xi} t^{-\frac{\alpha}{2\beta}} p(1, t^{-\frac{\alpha}{2\beta}} x) dx \\ &= \int_{\mathbb{R}} e^{-it^\alpha/2^\beta x \xi} p(1, x) dx = \mathcal{F} \{p(1, \cdot)\}(t^{\frac{\alpha}{2\beta}} \xi). \end{aligned}$$

Then by the Riemann-Lebesgue lemma,  $\mathcal{F} \{p(t, \cdot)\}(\xi)$  converges to 0 as  $t \rightarrow \infty$ . This implies  $R(\xi) \equiv 0$  and we obtain the desired result.

**Case 4:**  $d = 1$ ,  $\gamma = \beta$ . Now we additionally assume  $\sigma + \alpha \in \mathbb{N}$  and take  $\ell_0$  so that

$$\max(-2, -1 - \frac{1}{2\beta}) < \ell_0 < 0.$$

Note that every argument in Case 2 holds except that

$$\frac{\mathcal{H}_{\sigma,\beta}(z)}{2\beta(1+z)} = -\frac{\Gamma(\frac{1}{2} + \beta + \beta z)\Gamma(2+z)\Gamma(-z)}{2\Gamma(1-\beta-\beta z)\Gamma(1-\sigma+\alpha z)(z+1)}$$

has a removable singularity at  $z = -1$  if  $\sigma + \alpha \in \mathbb{N}$ . Thus (6.6) holds and we obtain (6.1). The theorems are proved.

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